

QUANTUM DIFFERENTIALS AND THE q -MONOPOLE REVISITED¹

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Abstract The q -monopole bundle introduced previously is extended to a general construction for quantum group bundles with non-universal differential calculi. We show that the theory applies to several other classes of bundles as well, including bicrossproduct quantum groups, the quantum double and combinatorial bundles associated to covers of compact manifolds.

1 Introduction

A ‘quantum group gauge theory’ in the sense of bundles with total and base ‘spaces’ noncommutative algebras (and quantum gauge group) has been introduced in [1] with the construction of the q -monopole over the q -sphere. Two nontrivial features of this q -monopole are the use of non-universal quantum differential calculi and construction in terms of patching of trivial bundles. Several aspects of general formalism concerning nonuniversal calculi were left open, however, and in the present paper we study some of these aspects further, providing a continuation of the general theory in [1].

We recall that in noncommutative geometry the nonuniqueness of the differential calculus is much more pronounced than it is classically. Although every algebra has a universal or ‘free’ calculus it is much too large and one has to quotient it if one is to have quantum geometries ‘deforming’ the classical situation. There are many ways to do this, however, and even for quantum groups (where we can demand (bi)covariance) the calculus is far from unique. In

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the case of quantum principal bundles one needs quantum differential calculi both on the base and on the quantum group fibre which have to fit together to provide a nontrivial calculus on the total space. This is the problem which we address here and its solution is the main result of the present paper. We introduce in Section 3 a natural construction which builds up the calculus on the total space of the bundle from specified ‘horizontal forms’ related to the base, a specified bicovariant calculus on the quantum group fibre and a connection form on the bundle with the universal calculus. Roughly speaking, it is the maximal differential calculus having the prescribed horizontal and fibre parts and such that the connection form is differentiable. This approach appears to be different from and, we believe, more complete than recent attempts on this problem in [2][3].

The remainder of the paper is devoted to examples and applications of this construction. We re-examine the q-monopole in Section 4 and verify that this example from [1] fits into the general formalism.

In Section 5 we consider a different application of the theory. We show that the combinatorial data associated to a cover of a compact manifold may be encoded in a discrete quantum differential calculus over the indexing set of the cover. This demonstrates the novel idea of doing (quantum) geometry of the combinatorics associated to a manifold rather than the combinatorics of the classical geometry. We show that the Czech cohomology may be recovered as the quantum cohomology over the cover. We also consider quantum group gauge theory over the cover as a potential source of new invariants of manifolds. Note that classical differential calculi are not possible over discrete sets, but nontrivial quantum ones are, i.e. this is a natural use of quantum geometry.

In Section 6 we further apply the theory to construct left-covariant quantum differential calculi on certain Hopf algebras of cross product form. We regard them as trivial quantum principal bundles and apply the results of Section 3. Examples include all cross product Hopf algebras such as the bicrossproduct quantum groups in [4], the biproducts and bosonisations [5][6] and the quantum double [7]. Although the bundles here are ‘trivial’, the uniform construction of natural quantum differential calculi on them by abstract methods would be a first step towards

their patching to obtain nontrivial bundles.

We begin the paper in Section 2 with some preliminaries from [1][8][9], including the definition of a trivial quantum principal bundle. This is basically an algebra factorising as $P = MH$ where M is the ‘base’ algebra and H is a quantum group. All algebras in the paper should be viewed as ‘coordinates’ although, when the algebra is noncommutative, they will not be the actual coordinate ring of any usual manifold. For quantum groups, we use the notations and conventions in [10]. In particular, $\Delta : H \rightarrow H \otimes H$ denotes the coproduct expressing the ‘group structure’ of quantum group H . $S : H \rightarrow H$ denotes the antipode expressing ‘group inversion’, and $\epsilon : H \rightarrow \mathbb{C}$ denotes the counit, expressing ‘evaluation at the group identity’. We work over \mathbb{C} . All general constructions not involving $*$ work over a general field just as well.

2 Preliminaries

In this section, we recall the basic definitions and notations to be used throughout the paper, up to and including the definition of a quantum principal bundle with nonuniversal calculus from [1]. The same formalism has been extended to braided group fibre and, beyond, to merely a coalgebra as fibre of the principal bundle[11], to which some of the results in the paper should extend.

If P is an algebra, we denote by $\Omega^1 P$ its universal or Kähler differential structure or quantum cotangent space. Here $\Omega^1 P = \ker \mu \subset P \otimes P$, where μ is the product map. The differential $d_U : P \rightarrow \Omega^1 P$ is $d_U u = 1 \otimes u - u \otimes 1$. We denote by $\Omega^1(P)$ a general nonuniversal differential structure or cotangent space. By definition, this is a P -bimodule and a map $d : P \rightarrow \Omega^1(P)$ obeying the Leibniz rule and such that $P \otimes P \rightarrow \Omega^1(P)$ provided by $u \otimes v \mapsto udv$ is surjective. It necessarily has the form $\Omega^1(P) = \Omega^1 P / \mathcal{N}$ where $\mathcal{N} \subset \Omega^1 P$ is a subbimodule, and $d = \pi_{\mathcal{N}} \circ d_U$ where $\pi_{\mathcal{N}}$ is the canonical projection. Nonuniversal calculi are in 1-1 correspondence with nonzero subbimodules \mathcal{N} .

When P is covariant under a quantum group H by a (say) right coaction $\Delta_R : P \rightarrow P \otimes H$ as a comodule algebra (i.e. Δ_R is a coaction and an algebra map), $\Omega^1(P)$ is right covariant (in an obvious way) iff $\Delta_R(\mathcal{N}) \subset \mathcal{N} \otimes H$. Here Δ_R is extended as the tensor product coaction to $P \otimes P$ and restricted to $\Omega^1 P$ for this equation to make sense. We will consider only calculi on

P of this form in the paper. Similar formulae hold for left covariance.

When H is a Hopf algebra the coproduct $\Delta : H \rightarrow H \otimes H$ can be viewed as both a right and a left coaction of H on itself by ‘translation’. We will be interested throughout in nonuniversal differential calculi $\Omega^1(H)$ which are both left and right covariant (i.e. bicovariant) under Δ . The subbimodule \mathcal{N} in the left covariant case is necessarily of the form $\mathcal{N} = \theta(H \otimes \mathcal{Q})$ where $\theta : H \otimes H \rightarrow H \otimes H$ is defined by

$$\theta(g \otimes h) = gSh_{(1)} \otimes h_{(2)} \quad (1)$$

and $\mathcal{Q} \subset \ker \epsilon \subset H$ is a right ideal. Left covariant calculi are in 1-1 correspondence with such \mathcal{Q} [12]. Bicovariant calculi $\Omega^1(H)$ are in 1-1 correspondence with right ideals \mathcal{Q} which are in addition stable under Ad in the sense $\text{Ad}(\mathcal{Q}) \subset \mathcal{Q} \otimes H$ [12]. Here Ad is the right adjoint coaction $\text{Ad}(h) = h_{(2)} \otimes (Sh_{(1)})h_{(3)}$. We use in these formulae the notation $\Delta h = h_{(1)} \otimes h_{(2)}$ (summation understood) of the resulting element of $H \otimes H$, and higher numbers for iterated coproducts. The universal calculus on H is bicovariant and corresponds to $\mathcal{Q} = \{0\}$.

The space $\ker \epsilon / \mathcal{Q}$ is the space of left-invariant 1-forms on H . We denote by $\pi_{\mathcal{Q}}$ the canonical projection. The dual of $\ker \epsilon / \mathcal{Q}$ (suitably defined) is the space of left-invariant vector fields or ‘invariant quantum tangent space’ on H . Hence a map which classically has values in the Lie algebra of gauge group will be formulated now as a map from $\ker \epsilon / \mathcal{Q}$. This is the approach in [1] for connections with nonuniversal calculi. Note that it depends on the choice of calculus. The moduli of bicovariant calculi (or more precisely, of quantum tangent spaces) on a general class of quantum groups has been obtained in [13]; it is typically discrete but infinite.

Since a general differential calculus is the projection of a universal one, it is natural to consider principal bundles and gauge theory with the universal calculi $\Omega^1 P$, $\Omega^1 H$ first, and construct the general bundles by making quotients. Therefore, we recall first the definitions for this universal case. A quantum principal H -bundle with the universal calculus is an H -covariant algebra P as above, such that the map $\chi : \Omega^1 P \rightarrow P \otimes \ker \epsilon$ defined by $\chi(u \otimes v) = u\Delta_R v$ is surjective and obeys $\ker \chi = P(\Omega^1 M)P$, where $M = \{u \in P | \Delta_R u = u \otimes 1\}$ is the invariant subalgebra. The latter plays the role of coordinates of the ‘base’. For a complete theory, we also require that P is flat as an M -bimodule. The surjectivity of χ corresponds in the geometric

case to the action being free. The kernel condition says that the joint kernel of all ‘left-invariant vector fields generated by the action’ (the maps $\Omega^1 P \rightarrow P$ obtained by evaluating against any element of $\ker \epsilon^*$) coincides with the ‘horizontal 1-forms’ $P(\Omega^1 M)P$ pulled back from the base. It plays the role in the proofs in [1] played classically by local triviality and dimensional arguments. The surjectivity and kernel conditions are equivalent to $\chi_M : P \otimes_M P \rightarrow P \otimes H$ being a bijection, where χ descends to the map χ_M (cf. [8, Proposition 1.6], [9, Lemma 3.2]). This is the Galois condition arising independently in a more algebraic context, cf [14] (not connected with connections and differential structures, however). We prefer to list the two conditions separately for conceptual reasons.

A connection ω_U on a quantum principal bundle with universal calculus is a map $\omega_U : \ker \epsilon \rightarrow \Omega^1 P$ such that $\chi \circ \omega_U = 1 \otimes \text{id}$ and $\Delta_R \circ \omega_U = (\omega_U \otimes \text{id}) \circ \text{Ad}$. It is shown in [1] that such connections are in 1-1 correspondence with equivariant complements to the horizontal forms $P(\Omega^1 M)P \subset \Omega^1 P$. We are now ready for the general case:

Definition 2.1 [1] *A general quantum principal bundle $P(M, H, \mathcal{N}, \mathcal{Q})$ is an H -covariant algebra P , an H -covariant calculus $\Omega^1(P)$ described by subbimodule \mathcal{N} and a bicovariant calculus $\Omega^1(H)$ described by Ad -invariant right ideal \mathcal{Q} compatible in the sense $\chi(\mathcal{N}) \subseteq P \otimes \mathcal{Q}$ and such that the map $\chi_{\mathcal{N}} : \Omega^1(P) \rightarrow P \otimes \ker \epsilon / \mathcal{Q}$ defined by $\chi_{\mathcal{N}} \circ \pi_{\mathcal{N}} = (\text{id} \otimes \pi_{\mathcal{Q}}) \circ \chi$ is surjective and has kernel $P(dM)P$.*

The surjectivity and kernel conditions here can also be written as an exact sequence

$$0 \rightarrow P(dM)P \rightarrow \Omega^1(P) \xrightarrow{\chi_{\mathcal{N}}} P \otimes \ker \epsilon / \mathcal{Q} \rightarrow 0, \quad (2)$$

and thus combined into single ‘differential Galois’ condition by noting that $\chi_{\mathcal{N}}$ descends to a map $\Omega^1(P)/P(dM)P \rightarrow P \otimes \ker \epsilon / \mathcal{Q}$ and requiring this to be an isomorphism. The condition $\chi(\mathcal{N}) \subseteq P \otimes \mathcal{Q}$ expresses ‘smoothness’ of the action and is needed for $\chi_{\mathcal{N}}$ to be well-defined. In fact if $P(M, H, \mathcal{N}, \mathcal{Q})$ is a quantum principal bundle then the inclusion above implies the equality $\chi(\mathcal{N}) = P \otimes \mathcal{Q}$ [8, Corollary 1.3]. On the other hand if $P(M, H)$ is already a quantum principal bundle with the universal calculus then the equality $\chi(\mathcal{N}) = P \otimes \mathcal{Q}$ is sufficient to ensure that $P(M, H, \mathcal{N}, \mathcal{Q})$ is a quantum principal bundle with the corresponding non-universal differential

calculi. Conversely, if $P(M, H, \mathcal{N}, \mathcal{Q})$ is a quantum principal bundle with the corresponding non-universal differential calculi then $P(M, H)$ is a quantum principal bundle with the universal calculus if and only if $\ker \chi \cap \mathcal{N} \subseteq P(\Omega^1 M)P \cap \mathcal{N}$ [15]. Finally, a connection on $P(M, H, \mathcal{N}, \mathcal{Q})$ is a map $\omega : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(P)$ such that $\chi_{\mathcal{N}} \circ \omega = 1 \otimes \text{id}$ and $\Delta_R \circ \omega = (\omega \otimes \text{id}) \circ \text{Ad}$. The Ad here denotes the quotient of the right adjoint coaction on H to the space $\ker \epsilon / \mathcal{Q}$ given by $\text{Ad} \circ \pi_{\mathcal{Q}} = (\pi_{\mathcal{Q}} \otimes \text{id}) \circ \text{Ad}$. As explained in [1], connections are in 1-1 correspondence with equivariant complements to the horizontal forms $P(dM)P \subset \Omega^1(P)$. See [1][9][16] for further details and formalism in this approach.

There are also two main general constructions for bundles and connections in [1], the first of them used to construct the local patches of the q -monopole and the second of them used to construct the q -monopole globally.

Example 2.2 [1] *Let P be an H -covariant algebra and suppose $\Phi : H \rightarrow P$ is a convolution-invertible linear map such that $\Phi(1) = 1$ and $\Delta_R \circ \Phi = (\Phi \otimes \text{id}) \circ \Delta$. Then $M \otimes H \rightarrow P$ by $m \otimes h \mapsto m\Phi(h)$ is a linear isomorphism and $P(M, H, \Phi)$ is a quantum principal bundle with universal calculus. There is a connection*

$$\omega_U(h) = \Phi^{-1}(h_{(1)})\beta_U(\pi_\epsilon(h_{(2)}))\Phi(h_{(3)}) + \Phi^{-1}(h_{(1)})d_U\Phi(h_{(2)}) \quad (3)$$

for any $\beta_U : \ker \epsilon \rightarrow \Omega^1 M$. Here $\pi_\epsilon(h) = h - \epsilon(h)$ is the projection to $\ker \epsilon$. The case $\beta = 0$ is called the trivial connection.

In fact, P is a cleft extension of M by H and has the structure of a cocycle cross product. If, in addition, \mathcal{Q} and \mathcal{N} define $\Omega^1(H)$ and $\Omega^1(P)$ as in Definition 2.1 then $P(M, H, \mathcal{N}, \mathcal{Q})$ is a quantum principal bundle with nonuniversal calculus. We call this a *trivial quantum principal bundle with general differential calculus*. We will obtain in the paper the construction of connections ω from β in this case.

Example 2.3 [1] *If P is itself a Hopf algebra and $\pi : P \rightarrow H$ a Hopf algebra surjection. P becomes H -covariant by $\Delta_R = (\text{id} \otimes \pi) \circ \Delta$. Suppose that the product map $\ker \pi|_M \otimes P \rightarrow \ker \pi$ is surjective. Then $P(M, H, \pi)$ is a quantum principal bundle with universal calculus. If there*

is a linear map $i : \ker \epsilon_H \rightarrow \ker \epsilon_P$ such that $\pi \circ i = \text{id}$ and $(\text{id} \otimes \pi) \circ \text{Ad} \circ i = (i \otimes \text{id}) \circ \text{Ad}$, then there is a connection

$$\omega_U(h) = (Si(h)_{(1)})di(h)_{(2)} \quad (4)$$

It is called the canonical connection associated to a linear splitting i .

Remark 2.4 Note that if P and H are Hopf algebras and $\pi : P \rightarrow H$ is a Hopf algebra surjection, then the canonical map χ is surjective since it is obtained by projecting the inverse θ^{-1} of the linear automorphism θ of $P \otimes P$ in (1) down to $P \otimes H$, i.e. $\chi = (\text{id} \otimes \pi) \circ \theta^{-1}$. The condition that the product map $\ker \pi|_M \otimes P \rightarrow \ker \pi$ be surjective provides that the kernel of χ is equal to horizontal one-forms. Combining [17, Theorem I] with [18, Lemma 1.3] one finds that $\ker \pi|_M \otimes P \rightarrow \ker \pi$ is surjective if there is a linear map $j : H \rightarrow P$ such that $j(1) = 1$ and $\Delta_R \circ j = (j \otimes \text{id}) \circ \Delta$. More precisely, [17, Theorem I] and [18, Lemma 1.3] imply that if such a j exists then in addition to $P(M, H, \pi)$ there is also a quantum principal bundle $P(M, H', \pi')$ where $H' = P/(\ker \pi|_M \cdot P)$. Therefore one can write the following commutative diagram

$$\begin{array}{ccccccc}
& 0 & \longrightarrow & 0 & \longrightarrow & \ker s & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P(\Omega^1 M)P & \longrightarrow & P \otimes P & \xrightarrow{(\text{id} \otimes \pi') \circ \theta^{-1}} & P \otimes H' \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow s & \\
0 & \longrightarrow & P(\Omega^1 M)P & \longrightarrow & P \otimes P & \xrightarrow{(\text{id} \otimes \pi) \circ \theta^{-1}} & P \otimes H \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & \longrightarrow & 0 & \longrightarrow & \text{cokers} &
\end{array}$$

The second and third row are exact by definition of a quantum principal bundle. Obviously $\text{cokers} = 0$. The application of the snake lemma (cf. [19, Section 1.2]) yields $\ker s = 0$, i.e. $H' \subseteq H$. Since $H = P/\ker \pi$, $H' = P/(\ker \pi|_M P)$ this implies that the product map $\ker \pi|_M \otimes P \rightarrow \ker \pi$ is surjective as required.

If there exists a left integral on H , i.e. $\lambda \in H^*$ such that $\lambda(1) = 1$ and $(\lambda \otimes \text{id}) \circ \Delta = \lambda$ then the map $j : H \rightarrow P$ can be defined by $j = \eta_P \circ \lambda$, where $\eta_P : \mathbb{C} \rightarrow P$ is the unit map. The map j is clearly an intertwiner since

$$\Delta_R j(h) = \lambda(h)1 \otimes 1 = \lambda(h_{(1)}) \otimes h_{(2)} = (j \otimes \text{id})\Delta(h).$$

In particular, if H is a compact quantum group in the sense of [20] then λ is the Haar measure on H . Therefore if H is a compact quantum group then the Hopf algebra surjection $\pi : P \rightarrow H$ leads immediately to the bundle $P(M, H, \pi)$. This fact is also proven directly by using representation theory of compact quantum groups in [21].

In the situation of Example 2.3, if $\Omega^1(P)$ is left covariant with its corresponding right ideal $\mathcal{Q}_P \subseteq \ker \epsilon \subset P$ obeying $(\text{id} \otimes \pi) \circ \text{Ad}(\mathcal{Q}_P) \subset \mathcal{Q}_P \otimes H$, then $\Omega^1(H)$ defined by $\mathcal{Q} = \pi(\mathcal{Q}_P)$ provides a quantum principal bundle $P(M, H, \pi, \mathcal{Q}_P)$. We call it a *homogeneous space bundle with general differential calculus*. If $i : \ker \epsilon_H \rightarrow \ker \epsilon_P$ is as above and, in addition, $i(\mathcal{Q}) \subset \mathcal{Q}_P$ then $\omega(h) = (Si(h)_{(1)})di(h)_{(2)}$ is a connection. A refinement of this construction will be provided in the paper.

3 Differential Calculi on Quantum Principal Bundles

In this section we obtain the main tool in the paper. This is a new construction for general quantum principal bundles with nonuniversal calculi, starting with a specified bicovariant calculus $\Omega^1(H)$ on the fibre and a specified ‘horizontal calculus’ on the base. In the classical case one has local triviality and one accordingly takes the calculus on P coinciding with its direct product form over each open set. That this is actually the standard calculus on P is consequence of the smoothness part of the axiom of local triviality. This is our motivation now.

As recalled in the Preliminaries, in the quantum case we actually have global conditions playing the role of local triviality[1], which is the ‘global approach’ which we describe first. Building up the calculus on P globally in this way means that we construct $\Omega^1(P)$ as the direct sum of a part from the base and a part from the fibre, i.e. actually the same process as building a connection ω . Therefore, the nonuniversal bundles constructed in this way will automatically have the property of existence of a natural connection.

On the other hand, the data going into the construction of $\Omega^1(P)$ should not already assume the existence of a bundle, as this is to be constructed. Instead, the additional input data besides the desired calculi $\Omega^1(H)$ and on the base should be related to the ‘topological’ and not ‘differential’ splitting. We therefore take for this additional ‘gluing’ datum a connection ω_U on

P as a quantum principal bundle with the universal calculus.

Accordingly, we let $P(M, H)$ be a quantum principal bundle with the universal calculus and $\Omega^1(H)$ a choice of bicovariant calculus on H defined by $\mathcal{Q} \subseteq \ker \epsilon \subset H$. As far as the differential calculus on the base is concerned, we can specify $\Omega^1(M)$ by $\mathcal{N}_M \subset \Omega^1 M$ as an M -subbimodule. More natural (and slightly more general) is to specify a ‘horizontal’ subbimodule \mathcal{N}_{hor} .

Lemma 3.1 *Let $P(M, H)$ be a quantum principal bundle with the universal calculus and let ω_U be a connection on it. Let \mathcal{Q} specify a bicovariant calculus on $\Omega^1(H)$. Let $h_\omega : P \otimes \mathcal{Q} \otimes P \rightarrow \Omega^1 P$ be a linear map given by $h_\omega(u, q, v) = uv^{(\bar{1})}\omega_U(qv^{(\bar{2})}) - u\omega_U(q)v$, where we write $\Delta_R u = u^{(\bar{1})} \otimes u^{(\bar{2})}$ (summation understood). Then $\mathcal{N}_0 = \text{Im} h_\omega \subset P(\Omega^1 M)P$ is a P -subbimodule invariant under Δ_R in the sense $\Delta_R \mathcal{N}_0 \subset \mathcal{N}_0 \otimes H$.*

Proof Clearly $wh_\omega(u, q, v) = h_\omega(wu, q, v)$, for any $u, v, w \in P$ and $q \in \mathcal{Q}$. Also

$$\begin{aligned} h_\omega(u, q, v)w &= uv^{(\bar{1})}\omega_U(qv^{(\bar{2})})w - u\omega_U(q)vw \\ &= uv^{(\bar{1})}\omega_U(qv^{(\bar{2})})w - uv^{(\bar{1})}w^{(\bar{1})}\omega_U(qv^{(\bar{2})}w^{(\bar{2})}) + h_\omega(u, q, vw) \\ &= h_\omega(uv^{(\bar{1})}, qv^{(\bar{2})}, w) + h_\omega(u, q, vw). \end{aligned}$$

Therefore $\mathcal{N}_0 = \text{Im} h_\omega$ is a subbimodule of $\Omega^1 P$. Furthermore

$$\chi(h_\omega(u, q, v)) = \chi(uv^{(\bar{1})}\omega_U(qv^{(\bar{2})}) - u\omega_U(q)v) = uv^{(\bar{1})} \otimes qv^{(\bar{2})} - (u \otimes q)(v^{(\bar{1})} \otimes v^{(\bar{2})}) = 0,$$

i.e. $\mathcal{N}_0 \in \ker \chi = P(\Omega^1 M)P$. Finally,

$$\begin{aligned} \Delta_R(h_\omega(u, q, v)) &= u^{(\bar{1})}v^{(\bar{1})}\omega_U(q_{(2)}v^{(\bar{2})}_{(3)}) \otimes u^{(\bar{2})}v^{(\bar{2})}_{(1)}Sv^{(\bar{2})}_{(2)}Sq_{(1)}q_{(3)}v^{(\bar{2})}_{(4)} \\ &\quad - u^{(\bar{1})}v^{(\bar{1})}\omega_U(q_{(2)}) \otimes u^{(\bar{2})}Sq_{(1)}q_{(3)}v^{(\bar{2})} \\ &= u^{(\bar{1})}v^{(\bar{1})}\omega_U(q_{(2)}v^{(\bar{2})}_{(1)}) \otimes u^{(\bar{2})}Sq_{(1)}q_{(3)}v^{(\bar{2})}_{(2)} \\ &\quad - u^{(\bar{1})}v^{(\bar{1})}\omega_U(q_{(2)}) \otimes u^{(\bar{2})}Sq_{(1)}q_{(3)}v^{(\bar{2})} \\ &= h_\omega(u^{(\bar{1})}, q_{(2)}, v^{(\bar{1})}) \otimes u^{(\bar{2})}Sq_{(1)}q_{(3)}v^{(\bar{2})} \in \mathcal{N}_0 \otimes H \end{aligned}$$

where we used the covariance of ω_U and the fact that \mathcal{Q} is Ad-invariant. Therefore \mathcal{N}_0 is right-invariant as stated. \square

We call any Δ_R -invariant P -subbimodule of $P(\Omega^1 M)P$ ‘horizontal’. We fix one, denoted \mathcal{N}_{hor} . The corresponding quotient $\Omega_{\text{hor}}^1 = P(\Omega^1 M)P/\mathcal{N}_{\text{hor}}$ is our choice of ‘horizontal’ part of the desired calculus on P .

Theorem 3.2 *Let $P(M, H)$, ω_U be a quantum principal bundle with the universal calculus and connection as above. Let \mathcal{Q} specify $\Omega^1(H)$ and*

$$\mathcal{N}_0 \subseteq \mathcal{N}_{\text{hor}} \subseteq P(\Omega^1 M)P$$

specify Ω_{hor}^1 . Then

$$\mathcal{N} = \langle \mathcal{N}_{\text{hor}}, P\omega_U(\mathcal{Q})P \rangle$$

specifies a differential calculus $\Omega^1(P)$ with the property that $P(M, H, \mathcal{N}, \mathcal{Q})$ is a quantum principal bundle, $P(dM)P = \Omega_{\text{hor}}^1$ and $\omega = \pi_{\mathcal{N}} \circ \omega_U$ is a connection on the bundle.

The calculus resulting from the choice $\mathcal{N}_{\text{hor}} = \mathcal{N}_0$ is called the maximal differential calculus compatible with ω_U . The choice $\mathcal{N}_{\text{hor}} = P(\Omega^1 M)P$ is called the minimal differential calculus compatible with ω_U .

Proof By assumption, $\Delta_R \mathcal{N}_{\text{hor}} \subset \mathcal{N}_{\text{hor}} \otimes H$. Also

$$\Delta_R(u\omega_U(q)v) = u^{(\bar{1})}\omega_U(q)^{(\bar{1})}v^{(\bar{1})} \otimes u^{(\bar{2})}\omega_U(q)^{(\bar{2})}v^{(\bar{2})} = u^{(\bar{1})}\omega_U(q_{(1)})v^{(\bar{1})} \otimes u^{(\bar{2})}(Sq_{(1)})q_{(3)}v^{(\bar{2})}$$

for all $u, v \in P$ and $q \in H$. The result is manifestly in $P\omega_U(\mathcal{Q})P \otimes H$ since \mathcal{Q} is Ad-invariant. Hence $\Delta_R \mathcal{N} \subset \mathcal{N} \otimes H$.

Next, we clearly have $\chi(\mathcal{N}_{\text{hor}}) = 0$. Then $\chi(u\omega_U(q)v) = u\chi(\omega_U(q))\Delta_R v = uv^{(\bar{1})} \otimes qv^{(\bar{2})} \in P \otimes \mathcal{Q}$ since \mathcal{Q} is a right ideal. Conversely, if $u \otimes q \in P \otimes \mathcal{Q}$ then $u\omega_U(q) \in P\omega_U(\mathcal{Q})P$ and $\chi(u\omega_U(q)) = u \otimes q$. Hence $\chi(\mathcal{N}) = P \otimes \mathcal{Q}$. We therefore have quantum principal bundle with nonuniversal differential calculus $\Omega^1(P)$.

Clearly $\pi_{\mathcal{N}} \circ \omega_U(\mathcal{Q}) = 0$, hence this descends to a map $\omega : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(P)$. Moreover,

$$\chi_{\mathcal{N}} \circ \pi_{\mathcal{N}} \circ \omega_U = (\text{id} \otimes \pi_{\mathcal{Q}}) \circ \chi \circ \omega_U = 1 \otimes \pi_{\mathcal{Q}}$$

on $\ker \epsilon$, where the first equality is the definition of $\chi_{\mathcal{N}}$ and the second is the equivariance of ω_U . Also,

$$\Delta_R \circ \pi_{\mathcal{N}} \circ \omega_U = (\pi_{\mathcal{N}} \otimes \text{id}) \circ \Delta_R \omega_U = (\pi_{\mathcal{N}} \circ \omega_U \otimes \text{id}) \circ \text{Ad}.$$

The first equality is clear from the definition of Δ_R . Hence we have a connection ω .

Finally, we note that the stated $\Omega^1(P)$ is uniquely determined by ω_U and \mathcal{N}_{hor} as the universal calculus with the stated properties. Thus, suppose that \mathcal{N}' defines another quantum differential calculus on P such that $\pi_{\mathcal{N}'} \circ \omega_U$ is a connection. Then $\omega_U(\mathcal{Q}) \subset \mathcal{N}'$. The stated \mathcal{N} is clearly the minimal subbimodule containing $P\omega_U(\mathcal{Q})P$ and \mathcal{N}_{hor} , i.e. any other such $\Omega^1(P)$ is a quotient. \square

There is a natural generalisation of this theorem in which we assume only that P is an H -comodule algebra (i.e. without going through the assumption that $P(M, H)$ is already a quantum principal bundle with the universal calculus). For this version we assume the existence of an Ad -equivariant map $\omega_U : \ker \epsilon \rightarrow \Omega^1 P$ obeying $\chi \circ \omega_U = 1 \otimes \text{id}$ and $\mathcal{Q}, \mathcal{N}_{\text{hor}}$ as above. Then the map $\chi_{\mathcal{N}}$ can be defined and if its kernel is $P(dM)P$ then the same conclusion holds.

We now consider how our construction looks for the two examples of quantum principal bundles with the universal calculus in the Preliminaries section.

Proposition 3.3 *Consider a trivial quantum principal bundle $P(M, H, \Phi)$ with the universal calculus and let $\Omega^1(H)$ and $\Omega^1(M)$ be determined by \mathcal{Q} and \mathcal{N}_M . Then for any $\beta_U : \ker \epsilon \rightarrow \Omega^1 M$ there is a differential calculus $\Omega^1(P)$ with $\Omega_{\text{hor}}^1 = P(dM)P$ and forming a trivial quantum principal bundle, and*

$$\omega(h) = \Phi^{-1}(h_{(1)})\beta \circ \pi_{\epsilon}(h_{(2)})\Phi(h_{(3)}) + \Phi^{-1}(h_{(1)})d\Phi(h_{(2)}) \quad (5)$$

is a connection on it for $\beta : \ker \epsilon \rightarrow \Omega_P^1(M)$, where obtained by restricting $\Omega_P^1(M) = \pi_{\mathcal{N}}(\Omega^1 M)$.

Proof We define $\omega_U : \ker \epsilon \rightarrow \Omega^1 P$ by $\omega_U(h) = \Phi^{-1}(h_{(1)})\beta_U(\pi_{\epsilon}(h_{(2)}))\Phi(h_{(3)}) + \Phi^{-1}(h_{(1)})d_U\Phi(h_{(2)})$ as a connection on the bundle with universal calculus. We also take $\mathcal{N}_{\text{hor}} = \langle P\mathcal{N}_M P, \mathcal{N}_0 \rangle$ where \mathcal{N}_0 is determined by ω_U . We can now apply Theorem 3.2. Note also that $\mathcal{N} = \langle P\mathcal{N}_M P, P\omega_U(\mathcal{Q})P \rangle$ as $\mathcal{N}_0 \subset P\omega_U(\mathcal{Q})P$.

Explicitly, the sub-bimodule corresponding to $\Omega^1(P)$ is $\mathcal{N} = \langle \mathcal{N}_{\text{hor}}, P\hat{\Phi}(\mathcal{N}_H)P \rangle$ where $\mathcal{N}_H = \theta(H \otimes \mathcal{Q})$ and $\hat{\Phi} : H \otimes H \rightarrow \Omega^1 P$,

$$\hat{\Phi}(g \otimes h) = \Phi(gh_{(1)})\Phi^{-1}(h_{(2)})\beta_U(h_{(3)})\Phi(h_{(4)}) + \Phi(gh_{(1)})\Phi^{-1}(h_{(2)}) \otimes \Phi(h_{(3)})$$

for all $h, g \in H$. This makes it clear that we recover here the construction for nonuniversal trivial bundles in terms of a map $\hat{\Phi}$ in [22]. Note that $\hat{\Phi} \circ \theta(g \otimes h) = \hat{\Phi}(gSh_{(1)} \otimes h_{(2)}) = \Phi(g)\omega_U(h)$, for any $g, h \in H$. Note that the inherited differential structure on M , $\Omega_P^1(M) \subset \Omega^1(P)$ is smaller than the original $\Omega^1(M) = \Omega^1 M / \mathcal{N}_M$ unless $\mathcal{N}_0 \cap \Omega^1 M \subseteq \mathcal{N}_P$. \square

From the proof of Theorem 3.2 we see that the resulting trivial bundle with nonuniversal calculus is of the general type discussed after Example 2.2; we succeed by the above to put a general class of connections on it. Also note that we may take more general \mathcal{N}_{hor} and any $\beta_U : \ker \epsilon \rightarrow \Omega^1 M$ to arrive at some $\Omega^1(P), \omega$, though not necessarily of the form stated.

Lemma 3.4 *Let $P(M, H, \Phi, \mathcal{Q}, \mathcal{N})$ be a trivial quantum principal bundle with differential calculus determined by \mathcal{Q} and \mathcal{N} . Then $\beta : \ker \epsilon \rightarrow \Omega_P^1(M)$ defines a connection ω by (5) in Proposition 3.3. if and only if for all $q \in \mathcal{Q}$,*

$$\Phi^{-1}(q_{(1)})\beta(\pi_\epsilon(q_{(2)}))\Phi(q_{(3)}) = -\Phi^{-1}(q_{(1)})d\Phi(q_{(2)}). \quad (6)$$

Furthermore, if Φ is an algebra map then for all $h \in H$,

$$\Phi^{-1}(q_{(1)})\beta(\pi_\epsilon(q_{(2)}h))\Phi(q_{(3)}) = \epsilon(h)\Phi^{-1}(q_{(1)})\beta(\pi_\epsilon(q_{(2)}))\Phi(q_{(3)}).$$

Proof Requirement (6) is another way of expressing the fact that $\omega(q) = 0$ for all $q \in \mathcal{Q}$. Since \mathcal{Q} is a right ideal, condition (6) implies that

$$h_{(1)} \otimes \Phi^{-1}(q_{(1)}h_{(2)})\beta(\pi_\epsilon(q_{(2)}h_{(3)}))\Phi(q_{(3)}h_{(4)}) \otimes h_{(5)} = -h_{(1)} \otimes \Phi^{-1}(q_{(1)}h_{(2)})d\Phi(q_{(2)}h_{(3)}) \otimes h_{(4)}.$$

Applying $\Phi \otimes \text{id} \otimes \Phi^{-1}$ and multiplying we thus obtain

$$\begin{aligned} \Phi(h_{(1)})\Phi^{-1}(q_{(1)}h_{(2)})\beta(\pi_\epsilon(q_{(2)}h_{(3)}))\Phi(q_{(3)}h_{(4)})\Phi^{-1}(h_{(5)}) &= \\ &= -\Phi(h_{(1)})\Phi^{-1}(q_{(1)}h_{(2)})d\Phi(q_{(2)}h_{(3)})\Phi^{-1}(h_{(4)}). \end{aligned}$$

If Φ is an algebra map the above formula simplifies further

$$\Phi^{-1}(q_{(1)})\beta(\pi_\epsilon(q_{(2)}h))\Phi(q_{(3)}) = -\Phi^{-1}(q_{(1)})d(\Phi(q_{(2)})\Phi(h_{(1)}))\Phi^{-1}(h_{(2)}).$$

The application of the Leibniz rule and the fact that $q \in \ker \epsilon$ yields

$$\Phi^{-1}(q_{(1)})\beta(\pi_\epsilon(q_{(2)}h))\Phi(q_{(3)}h) = -\epsilon(h)\Phi^{-1}(q_{(1)})d\Phi(q_{(2)}),$$

which in view of (6) implies the assertion. Notice that the condition one obtains in this way deals entirely with the structure of \mathcal{N}_{hor} and is the consequence of the existence of \mathcal{N}_0 . \square

Proposition 3.5 *Consider a quantum principal bundle with the universal calculus of the homogeneous type $P(M, H, \pi)$ where $\pi : P \rightarrow H$ is a Hopf algebra surjection. For any $\Omega^1(H)$, if ω_U is left-invariant and \mathcal{N}_{hor} is left-invariant under the left-regular coaction of P as a Hopf algebra, then $\Omega^1(P)$ in Theorem 3.2 is left covariant. Moreover, left-invariant ω_U are canonical connections in 1-1 correspondence with i as in Example 2.3. Left-covariant \mathcal{N}_{hor} are in 1-1 correspondence with right ideals $\mathcal{Q}_0 \subset \mathcal{Q}_{\text{hor}} \subseteq \ker \pi$, where*

$$\mathcal{Q}_0 = \text{span}\{i(q)u - i(q\pi(u)) \mid q \in \mathcal{Q}, u \in P\}.$$

Proof We can regard \mathcal{N}_{hor} as a subbimodule of $\Omega^1 P$. As such, it defines a differential calculus $\Omega^1 P / \mathcal{N}_{\text{hor}}$ on P . As P is now a Hopf algebra, the calculus is left covariant i.e. $\Delta_L \mathcal{N}_{\text{hor}} \subset \mathcal{N}_{\text{hor}} \otimes P$ iff $\mathcal{N}_{\text{hor}} = \theta(P \otimes \mathcal{Q}_{\text{hor}})$ for a right ideal $\mathcal{Q}_{\text{hor}} \subseteq \ker \epsilon \subset P$. Here Δ_L is the left regular coaction or ‘translation’ on $P \otimes P$ obtained from the coproduct.

On the other hand, since $\mathcal{N}_{\text{hor}} \subset P(\Omega^1 M)P$, we know that $\chi(\mathcal{N}_{\text{hor}}) = 0$. Take $q \in \mathcal{Q}_{\text{hor}}$. Then $0 = \chi\theta(1 \otimes q) = (Sq_{(1)})q_{(2)} \otimes \pi(q_{(3)}) = \pi(q)$ so $\mathcal{Q}_{\text{hor}} \subseteq \ker \pi$. Conversely, if $\mathcal{Q}_{\text{hor}} \subseteq \ker \pi$ then clearly $\mathcal{N}_{\text{hor}} = \theta(P \otimes \mathcal{Q}_{\text{hor}}) \subset P(\Omega^1 M)P$, since $\ker \pi = (\ker \epsilon |_M)P$.

If the connection ω_U is invariant in the sense $\Delta_L \omega_U(h) = 1 \otimes \omega_U(h)$ for any $h \in \ker \epsilon$ then clearly $\Delta_L(u\omega_U(q)v) \in P \otimes P\omega_U(\mathcal{Q})P$ for all $u, v \in P$ and $q \in \mathcal{Q} \subset \ker \epsilon \subset H$. Therefore \mathcal{N} defined in Theorem 3.2 obeys $\Delta_L \mathcal{N} \subset P \otimes \mathcal{N}$, i.e. $\Omega^1(P)$ is left covariant.

The canonical connection associated to i as in Example 2.3 is invariant:

$$\begin{aligned} \Delta_L \omega_U(h) &= \Delta_L((Si(h)_{(1)})d_U i(h)_{(2)}) = (Si(h)_{(2)})i(h)_{(3)} \otimes Si(h)_{(1)} \otimes i(h)_{(4)} \\ &= 1 \otimes Si(h)_{(1)} \otimes i(h)_{(2)} = 1 \otimes \omega_U(h). \end{aligned}$$

Conversely if ω_U is an invariant connection, we define $i : \ker \epsilon_H \rightarrow \ker \epsilon_P$ by $i = (\epsilon_P \otimes \text{id}) \circ \omega_U$. Let θ^{-1} be the inverse to the canonical map $\theta : P \otimes P \rightarrow P \otimes P$ defined as in (1). Explicitly $\theta^{-1}(u \otimes v) = uv_{(1)} \otimes v_{(2)}$. Clearly $\theta^{-1} = (\text{id} \otimes \epsilon_P \otimes \text{id}) \circ \Delta_L$. Since ω is left-invariant one immediately finds that $\theta^{-1} \circ \omega_U(h) = 1 \otimes i(h)$. Thus $\omega_U(h) = \theta(1 \otimes i(h)) = Si(h)_{(1)} \otimes i(h)_{(2)} = Si(h)_{(1)} d_U i(h)_{(2)}$ and ω_U has the structure of the canonical connection associated to i . It remains to prove that i is an Ad-covariant splitting. Since $\chi = (\text{id} \otimes \pi) \circ \theta^{-1}$ the fact that $\chi \circ \omega_U(h) = 1 \otimes h$ implies that $\pi(i(h)) = h$. Finally compute

$$\Delta_R(\omega_U(h)) = Si(h)_{(2)} \otimes i(h)_{(3)} \otimes \pi(Si(h)_{(1)} i(h)_{(4)}).$$

On the other hand ω_U is a connection therefore

$$\Delta_R(\omega_U(h)) = \omega_U(h_{(2)}) \otimes Sh_{(1)} h_{(3)} = Si(h_{(2)})_{(1)} \otimes i(h_{(2)})_{(2)} \otimes Sh_{(1)} h_{(3)}.$$

Applying $(\epsilon_P \otimes \text{id} \otimes \text{id})$ to above equality one obtains the required Ad-covariance of i .

Using the fact that ω_U is left-invariant we find

$$\begin{aligned} \Delta_L(h_\omega(u, q, v)) &= u_{(1)} v_{(1)} \otimes u_{(2)} v_{(2)} \omega_U(q\pi(v_{(3)})) - u_{(1)} v_{(1)} \otimes u_{(2)} v_{(2)} \omega_U(q) \\ &= u_{(1)} v_{(1)} \otimes h_\omega(u_{(2)}, q, v_{(2)}), \end{aligned}$$

where h_ω is the map defined in Lemma 3.1. Therefore \mathcal{N}_0 is left-invariant and there is corresponding right ideal $\mathcal{Q}_0 \in \ker \epsilon_P$ given by $\mathcal{N}_0 = \theta(P \otimes \mathcal{Q}_0)$. Since \mathcal{N}_{hor} contains necessarily \mathcal{N}_0 , the right ideal \mathcal{Q}_{hor} must contain \mathcal{Q}_0 . For the canonical connection induced by the splitting i , \mathcal{Q}_0 comes out as stated. The fact that \mathcal{Q}_0 is a right ideal can be established directly since

$$(i(q)u - i(q\pi(u)))v = (i(q)uv - i(q\pi(uv))) - (i(q\pi(u))v - i(q\pi(u)\pi(v))) \in \mathcal{Q}_0.$$

For completeness, we also show that the resulting bundle is indeed of the natural nonuniversal homogeneous type discussed after Remark 2.4. First of all note that $\theta^{-1}(u\omega_U(q)v) = u(Si(q)_{(1)}i(q)_{(2)}v_{(1)} \otimes i(q)_{(3)}v_{(2)}) = uv_{(1)} \otimes i(q)v_{(2)}$. Hence $\mathcal{N} = \theta(P \otimes \mathcal{Q}_P)$ where $\mathcal{Q}_P = \langle \mathcal{Q}_{\text{hor}}, i(Q)P \rangle$. From this it is also clear that $\Omega^1(P)$ is left covariant, as \mathcal{Q}_P is clearly a right ideal. Also, $\pi(\mathcal{Q}_P) = \mathcal{Q}$. It remains to verify whether $(\text{id} \otimes \pi)\text{Ad}(\mathcal{Q}_P) \subset \mathcal{Q}_P \otimes H$. Take any $q \in \mathcal{Q}_P$, then $Sq_{(1)} \otimes q_{(2)} \in \mathcal{N}$. By construction $\Omega^1(P)$ is right H -covariant,

therefore $Sq_{(2)} \otimes q_{(3)} \otimes \pi(Sq_{(1)}q_{(4)}) \in \mathcal{N} \otimes H$. Applying $\theta^{-1} \otimes \text{id}$ to this one thus obtains that $1 \otimes q_{(2)} \otimes \pi(Sq_{(1)}q_{(3)}) \in P \otimes \mathcal{Q}_P \otimes H$. Therefore $(\text{id} \otimes \pi)\text{Ad}(\mathcal{Q}_P) \subset \mathcal{Q}_P \otimes H$ as required.

In this case it is clear that $i(\mathcal{Q}) \subset \mathcal{Q}_P$, i.e. the canonical connection is of the type mentioned after Remark 2.4 from [1]. \square

In the case of a homogeneous quantum principal bundle with a general differential calculus of the type discussed after Remark 2.4. we can establish the one-to-one correspondence between invariant connections and Ad-covariant splittings as follows. The conditions satisfied by \mathcal{Q}_P and \mathcal{Q} allow for definition of maps $\overline{\text{Ad}} : \ker \epsilon_P / \mathcal{Q}_P \rightarrow \ker \epsilon_P / \mathcal{Q}_P \otimes H$ and $\overline{\pi} : \ker \epsilon_P / \mathcal{Q}_P \rightarrow \ker \epsilon_H / \mathcal{Q}$ by $\overline{\text{Ad}} \circ \pi_{\mathcal{Q}_P} = (\pi_{\mathcal{Q}_P} \otimes \pi) \circ \text{Ad}$ and $\overline{\pi} \circ \pi_{\mathcal{Q}_P} = \pi_{\mathcal{Q}} \circ \pi$. Here $\pi_{\mathcal{Q}} : \ker \epsilon_H \rightarrow \ker \epsilon_H / \mathcal{Q}$ and $\pi_{\mathcal{Q}_P} : \ker \epsilon_P \rightarrow \ker \epsilon_P / \mathcal{Q}_P$ are canonical surjections.

Proposition 3.6 *The left-covariant connections ω in $P(M, H, \pi, \mathcal{Q}_P)$ are in one-to-one correspondence with the linear maps $i : \ker \epsilon_H / \mathcal{Q} \rightarrow \ker \epsilon_P / \mathcal{Q}_P$ such that $\overline{\pi} \circ i = \text{id}$ and $\overline{\text{Ad}} \circ i = (i \otimes \text{id}) \circ \text{Ad}$.*

Proof Assume that $\omega : \ker \epsilon_H / \mathcal{Q} \rightarrow \Omega^1(P)$ is an invariant connection in $P(M, H, \pi, \mathcal{Q}_P)$. Define a map $\overline{\epsilon} : \Omega^1(P) \rightarrow \ker \epsilon_P / \mathcal{Q}_P$ by the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \Omega^1 P & \xrightarrow{\pi \mathcal{N}} & \Omega^1(P) \longrightarrow 0 \\ & & \downarrow \epsilon_P \otimes \text{id} & & \downarrow \epsilon_P \otimes \text{id} & & \downarrow \overline{\epsilon} \\ 0 & \longrightarrow & \mathcal{Q}_P & \longrightarrow & \ker \epsilon_P & \xrightarrow{\pi \mathcal{Q}_P} & \ker \epsilon_P / \mathcal{Q}_P \longrightarrow 0 \end{array}$$

Let $i = \overline{\epsilon} \circ \omega$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \Omega^1 P & \xrightarrow{\pi \mathcal{N}} & \Omega^1(P) \\ & & \downarrow \Delta_L & & \downarrow \Delta_L & & \downarrow \Delta_L \\ 0 & \longrightarrow & P \otimes \mathcal{N} & \longrightarrow & P \otimes \Omega^1 P & \xrightarrow{\text{id} \otimes \pi \mathcal{N}} & P \otimes \Omega^1(P) \longrightarrow 0 \\ & & \downarrow \text{id} \otimes \epsilon_P \otimes \text{id} & & \downarrow \text{id} \otimes \epsilon_P \otimes \text{id} & & \downarrow \text{id} \otimes \overline{\epsilon} \\ 0 & \longrightarrow & P \otimes \mathcal{Q}_P & \longrightarrow & P \otimes \ker \epsilon_P & \xrightarrow{\text{id} \otimes \pi \mathcal{Q}_P} & P \otimes \ker \epsilon_P / \mathcal{Q}_P \longrightarrow 0 \\ & & \downarrow \text{id} \otimes \pi & & \downarrow \text{id} \otimes \pi & & \downarrow \text{id} \otimes \overline{\pi} \\ & & P \otimes \mathcal{Q} & \longrightarrow & P \otimes \ker \epsilon_H & \xrightarrow{\text{id} \otimes \pi \mathcal{Q}} & P \otimes \ker \epsilon_H / \mathcal{Q} \longrightarrow 0 \end{array} \tag{7}$$

The first two maps Δ_L are left coactions of P on $P \otimes P$ obtained from the left regular coaction of P provided by the coproduct while the third Δ_L is their projection to $\Omega^1(P)$. The third column gives the map χ , therefore the fourth column describes $\chi_{\mathcal{N}}$, i.e. $\chi_{\mathcal{N}} = (\text{id} \otimes \bar{\pi} \circ \bar{\epsilon}) \circ \Delta_L$. Since ω is a connection, $\chi_{\mathcal{N}}(\omega(h)) = 1 \otimes h$ for any $h \in \ker \epsilon_H / \mathcal{Q}$. Thus we have

$$1 \otimes h = (\text{id} \otimes \bar{\pi} \circ \bar{\epsilon}) \circ \Delta_L(\omega(h)) = 1 \otimes \bar{\pi} \circ \bar{\epsilon} \circ \omega(h) = 1 \otimes \bar{\pi} \circ i(h).$$

To derive the second equality we used invariance of ω . Therefore $\bar{\pi} \circ i = \text{id}$.

Next consider the map $\theta_N : \Omega^1(P) \rightarrow P \otimes \ker \epsilon_P / \mathcal{Q}_P$, given by $\theta_N = (\text{id} \otimes \bar{\epsilon}) \circ \Delta_L$. This map makes the following diagram commute

$$\begin{array}{ccccccc} & 0 & \longrightarrow & 0 & \longrightarrow & \ker \theta_N & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \Omega^1 P & \xrightarrow{\pi_{\mathcal{N}}} & \Omega^1(P) \longrightarrow 0 \\ & \downarrow \theta^{-1} & & \downarrow \theta^{-1} & & \downarrow \theta_N & \\ 0 & \longrightarrow & P \otimes \mathcal{Q}_P & \longrightarrow & P \otimes \ker \epsilon_P & \xrightarrow{\text{id} \otimes \pi_{\mathcal{Q}_P}} & P \otimes \ker \epsilon_P / \mathcal{Q}_P \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & \longrightarrow & 0 & \longrightarrow & \text{coker} \theta_N & \end{array}$$

This diagram is a combination of the first three rows of (7). Clearly $\text{coker} \theta_N = 0$. By the snake lemma (cf. [19, Section 1.2]), $\ker \theta_N = 0$. Therefore θ_N is a bijection. The left-invariance of ω implies that

$$\theta_N(\omega(h)) = (\text{id} \otimes \bar{\epsilon}) \circ \Delta_L(\omega(h)) = 1 \otimes \bar{\epsilon}(\omega(h)) = 1 \otimes i(h)$$

for any $h \in \ker \epsilon_H / \mathcal{Q}$. Therefore $\omega(h) = \theta_N^{-1}(1 \otimes i(h))$.

Using $\overline{\text{Ad}}$ and Δ_R one constructs the tensor product coaction $\overline{\Delta}_R : P \otimes \ker \epsilon_P / \mathcal{Q}_P \rightarrow P \otimes \ker \epsilon_P / \mathcal{Q}_P \otimes H$. Then θ_N is a right H -comodule isomorphism. This follows from the fact that θ^{-1} is a corresponding H -comodule isomorphism. Explicitly

$$\begin{aligned} \Delta_R(\theta^{-1}(u \otimes v)) &= \Delta_R(uv_{(1)} \otimes v_{(2)}) = u_{(1)}v_{(1)} \otimes v_{(4)} \otimes \pi(u_{(2)}v_{(2)}Sv_{(3)}v_{(5)}) \\ &= u_{(1)}v_{(1)} \otimes v_{(2)} \otimes \pi(u_{(2)}v_{(3)}). \end{aligned}$$

Δ_R here is a right coaction of H on $P \otimes \ker \epsilon_P$ built with $(\text{id} \otimes \pi) \circ \Delta$ on P and $(\text{id} \otimes \pi) \circ \text{Ad}$ on $\ker \epsilon_P$. On the other hand

$$(\theta^{-1} \otimes \text{id})(\Delta_R(u \otimes v)) = \theta^{-1}(u_{(1)} \otimes v_{(1)}) \otimes \pi(u_{(2)}v_{(2)}) = u_{(1)}v_{(1)} \otimes v_{(2)} \otimes \pi(u_{(2)}v_{(3)}),$$

where Δ_R is a standard tensor product coaction of H on $\Omega^1 P \subset P \otimes P$. Therefore

$$\begin{aligned}\overline{\Delta}_R \circ \theta_N \circ \pi_{\mathcal{N}} &= \overline{\Delta}_R \circ (\text{id} \otimes \pi_{\mathcal{Q}_P}) \circ \theta^{-1} = (\text{id} \otimes \pi_{\mathcal{Q}_P} \otimes \text{id}) \circ \Delta_R \circ \theta^{-1} \\ &= (\text{id} \otimes \pi_{\mathcal{Q}_P} \otimes \text{id}) \circ (\theta^{-1} \otimes \text{id}) \circ \Delta_R = (\theta_N \otimes \text{id}) \circ (\pi_{\mathcal{N}} \otimes \text{id}) \circ \Delta_R \\ &= (\theta_N \otimes \text{id}) \circ \Delta_R \circ \pi_{\mathcal{N}}.\end{aligned}$$

Therefore θ_N is an intertwiner as stated. Its inverse is also an intertwiner. We compute

$$\Delta_R \circ \theta_N^{-1}(1 \otimes i(h)) = (\theta_N^{-1} \otimes \text{id}) \circ \overline{\Delta}_R(1 \otimes i(h)) = (\theta_N^{-1} \otimes \text{id})(1 \otimes \overline{\text{Ad}}(i(h))).$$

On the other hand, since ω is a connection this is equal to $\Delta_R(\omega(h)) = (\omega \otimes \text{id}) \circ \text{Ad}(h)$. Applying $(\theta_N \otimes \text{id})$ to both sides one obtains

$$1 \otimes ((i \otimes \text{id}) \circ \text{Ad}(h)) = 1 \otimes \overline{\text{Ad}} \circ i(h),$$

i.e. $\overline{\text{Ad}} \circ i = (i \otimes \text{id}) \circ \text{Ad}$, as required.

Conversely, given $i : \ker \epsilon_H / \mathcal{Q} \rightarrow \ker \epsilon_P / \mathcal{Q}_P$ with the properties described in the proposition, one defines a map $\omega : \ker \epsilon_H / \mathcal{Q} \rightarrow \Omega^1(P)$ by $\omega(h) = \theta_N^{-1}(1 \otimes i(h))$. The Ad -covariance of i implies the Ad -covariance of ω since θ_N^{-1} is an intertwiner of Δ_R and $\overline{\Delta}_R$. Furthermore, since $\chi_{\mathcal{N}} = (\text{id} \otimes \overline{\pi}) \circ \theta_N$ from diagram (7),

$$\chi_{\mathcal{N}}(\omega(h)) = 1 \otimes \overline{\pi}(i(h)) = 1 \otimes h.$$

Therefore ω is a connection. The fact that ω obtained in this way is left-covariant is well-known from the theory of left-covariant calculi [12] but we include the proof for the completeness. First consider any $u \otimes v \in P \otimes P$ and compute

$$\Delta_L(\theta(u \otimes v)) = \Delta_L(u S v_{(1)} \otimes v_{(2)}) = u_{(1)} S v_{(2)} v_{(3)} \otimes u_{(2)} S v_{(1)} \otimes v_{(4)} = u_{(1)} \otimes \theta(u_{(2)} \otimes v).$$

This implies that

$$\Delta_L(\theta_N^{-1}(u \otimes v)) = u_{(1)} \otimes \theta_N^{-1}(u_{(2)} \otimes v).$$

for any $u \in P$ and $v \in \ker \epsilon_P / \mathcal{Q}_P$. Therefore

$$\Delta_L \omega(h) = \Delta_L \circ \theta_N^{-1}(1 \otimes i(h)) = 1 \otimes \theta_N^{-1}(1 \otimes i(h)) = 1 \otimes \omega(h),$$

for any $h \in \ker \epsilon_H / \mathcal{Q}$. This completes the proof. \square

Example 3.7 Consider a homogeneous quantum principal bundle $P(M, H, \pi)$ with the universal calculus and split by $i : \ker \epsilon_H \rightarrow \ker \epsilon_P$. Let $\Omega^1(H)$ and $\Omega^1(M)$ be determined by \mathcal{Q} and \mathcal{N}_M . Then there is a differential calculus $\Omega^1(P)$ with $\Omega_{\text{hor}}^1 = P(dM)P$ and $\omega(h) = (Si(h)_{(1)})di(h)_{(2)}$ is a connection on it. If $\Omega^1(M)$ is left P -covariant then $\Omega^1(P)$ is left-covariant. The corresponding canonical map $\ker \epsilon_H / \mathcal{Q} \rightarrow \ker \epsilon_P / \mathcal{Q}_P$ from Proposition 3.5 in this case is $[h] \mapsto \pi_{\mathcal{Q}_P} \circ i(h)$, where $h \in \pi_{\mathcal{Q}}^{-1}([h]) \subset \ker \epsilon_H$.

Proof We take $\mathcal{N}_{\text{hor}} = \langle P\mathcal{N}_M P, \mathcal{N}_0 \rangle$ as in Proposition 3.3. Then

$$\Delta_L(um \otimes nv) = u_{(1)}(m \otimes n)^{(\bar{1})}v_{(1)} \otimes u_{(2)}(m \otimes n)^{(\bar{2})}v \in P \otimes \mathcal{N}_{\text{hor}}$$

for all $u, v \in P$ and $m \otimes n \in \mathcal{N}_M$ provided $\Delta_L \mathcal{N}_M \subset P \otimes \mathcal{N}_M$. Therefore Ω_{hor}^1 is left-covariant and the assertion follows from Proposition 3.5. As in Proposition 3.3 the inherited $\Omega_P^1(M)$ is a quotient of $\Omega^1(M) = \Omega^1 M / \mathcal{N}_M$ unless $\mathcal{N}_0 \cap \Omega^1 M \subseteq \mathcal{N}_M$. \square

This provides a natural construction for homogeneous bundles (where P is a Hopf algebra) to have differential calculi which are left-covariant. We conclude with the simplest concrete example of our construction in Theorem 3.2.

Example 3.8 Let $P = H$ regarded as a trivial quantum principal bundle with $M = \mathbb{C}$ and the universal calculus. The trivialisation is $\Phi = \text{id}$ and the associated trivial connection is the unique nonzero ω_U . Hence, for every bicovariant $\Omega^1(H)$ Theorem 3.2 induces a natural Maurer-Cartan connection $\omega : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(H)$.

Proof Here $\Omega^1 M = 0$ so $\mathcal{N}_{\text{hor}} = 0$ and $\beta = 0$ is the only choice in Proposition 3.3. In fact, there is a unique connection ω_U since $\chi = \theta^{-1}$ so that the condition $\chi \circ \omega_U(h) = 1 \otimes h$ implies that $\omega_U(h) = \theta(1 \otimes h)$. This is the Maurer-Cartan form with the universal calculus. We then apply Theorem 3.2. \square

4 Differential Structures on the q -Monopole Bundle

Recall from [1] that the q -monopole (of charge 2) is a canonical connection in the bundle $SO_q(3)(S_q^2, \mathbb{C}[Z, Z^{-1}], \pi)$. The quantum group $SO_q(3)$ is a subalgebra of $SU_q(2)$ spanned by all

monomials of even degree. $SU_q(2)$ is generated by the identity and a matrix $\mathbf{t} = (t_{ij}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, subject to the homogeneous relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma, \quad \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma,$$

and a determinant relation $\alpha\delta - q\beta\gamma = 1$, $q \in \mathbb{C}^*$. We assume that q is not a root of unity. $SU_q(2)$ has a matrix quantum group structure,

$$\Delta t_{ij} = \sum_{k=1}^2 t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad S\mathbf{t} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}.$$

The structure quantum group of the q -monopole bundle is an algebra of functions on $U(1)$, i.e. the algebra $\mathbb{C}[Z, Z^{-1}]$ of formal power series in Z and Z^{-1} , where Z^{-1} is an inverse of Z . It has a standard Hopf algebra structure

$$\Delta Z^{\pm 1} = Z^{\pm 1} \otimes Z^{\pm 1}, \quad \epsilon(Z^{\pm 1}) = 1, \quad SZ^{\pm 1} = Z^{\mp 1}.$$

There is a Hopf algebra projection $\pi : SO_q(3) \rightarrow k[Z, Z^{-1}]$, built formally from $\pi_{\frac{1}{2}} : SU_q(2) \rightarrow \mathbb{C}[Z^{\frac{1}{2}}, Z^{-\frac{1}{2}}]$,

$$\pi_{\frac{1}{2}} : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} Z^{\frac{1}{2}} & 0 \\ 0 & Z^{-\frac{1}{2}} \end{pmatrix},$$

which defines a right coaction $\Delta_R : SO_q(3) \rightarrow SO_q(3) \otimes \mathbb{C}[Z, Z^{-1}]$ by $\Delta_R = (\text{id} \otimes \pi) \circ \Delta$. Finally $S_q^2 \subset SO_q(3)$ is a quantum two-sphere [23], defined as a fixed point subalgebra, $S_q^2 = SO_q(3)^{\mathbb{C}[Z, Z^{-1}]}$. S_q^2 is generated by $\{1, b_- = \alpha\beta, b_+ = \gamma\delta, b_3 = \alpha\delta\}$ and the algebraic relations in S_q^2 may be deduced from those in $SO_q(3)$.

The canonical connection in the q -monopole bundle ω_D is provided by the map $i : \mathbb{C}[Z, Z^{-1}] \rightarrow SO_q(3)$ given by $i(Z^n) = \alpha^{2n}$, $i(Z^{-n}) = \delta^{2n}$, $n = 0, 1, \dots$ (restricted to $\ker \epsilon_{\mathbb{C}[Z, Z^{-1}]}$). In this section we construct differential structures on the q -monopole bundle using ω_D .

Similarly as in [1] we choose a differential structure on $\mathbb{C}[Z, Z^{-1}]$ to be given by the right ideal \mathcal{Q} generated by $Z^{-1} + q^4 Z - (1 + q^4)$. The space $\ker \epsilon / \mathcal{Q}$ is one-dimensional and we denote by $[Z - 1]$ its basic element obtained by projecting $Z - 1$ down to $\ker \epsilon / \mathcal{Q}$.

Proposition 4.1 *Let for a quantum principal bundle $SO_q(3)(S_q^2, \mathbb{C}[Z, Z^{-1}], \pi)$, \mathcal{Q} and i be as above. Then the minimal horizontal ideal $\mathcal{Q}_0 \in \ker \epsilon_{SO_q(3)}$ defined in Proposition 3.5 is generated*

by the following elements of $\ker \pi$

$$\beta\gamma, \quad q^4\alpha^3\beta + \delta\beta - (1 + q^4)\alpha\beta, \quad q^4\alpha^3\gamma + \delta\gamma - (1 + q^4)\alpha\gamma$$

The space $\ker \pi / \mathcal{Q}_0$ and thus the corresponding differential calculus are infinite-dimensional.

Let $\mathcal{Q}^{(k,l)}$, $k, l = 1, 2, \dots$ be an infinite family of right ideals in $\ker \pi$ generated by the generators of \mathcal{Q}_0 and additionally by β^{2k} , γ^{2l} . For each pair (k, l) , $\ker \pi / \mathcal{Q}^{(k,l)}$ is $4(k + l - 1)$ -dimensional.

Furthermore let $\mathcal{Q}^{(k,l;r,s)}$, $k, l = 1, 2, \dots$, $r = 0, 1, \dots, k$, $s = 0, 1, \dots, l$ be an infinite family of right ideals in $\ker \pi$ generated by the generators of $\mathcal{Q}^{(k,l)}$ and also by $(\alpha - \delta)\beta^{2r-1}$, $(\alpha - \delta)\gamma^{2s-1}$. Then $\ker \pi / \mathcal{Q}^{(k,l;r,s)}$ is a $3k + 3l + r + s - 4$ -dimensional vector space. Notice also that $\mathcal{Q}^{(k,l;k,l)} = \mathcal{Q}^{(k,l)}$.

Proof The generators of \mathcal{Q}_0 are obtained by a direct computation of the ideal given in Proposition 3.5. Explicitly, $\beta\gamma$ is computed by taking $q = Z^{-1} + q^4Z - (1 + q^4)$ and $u = \alpha^2$ and $u = \delta^2$. The remaining two elements are obtained by taking $q = 1 + q^4Z^2 - (1 + q^4)Z$ and $u = \alpha\beta$ and $u = \alpha\gamma$ correspondingly. It can be then shown that all the other elements of \mathcal{Q}_0 are generated from the three listed in the proposition. For example, the choice $q = Z^{-2} + q^4 - (1 + q^4)Z^{-1}$ and $u = \delta\beta$ gives $\delta^3\beta + q^4\alpha\beta - (1 + q^4)\delta\beta$, but

$$\delta^3\beta + q^4\alpha\beta - (1 + q^4)\delta\beta = q^{-2}(q^4\alpha^3\beta + \delta\beta - (1 + q^4)\alpha\beta)\delta^2 - \beta\gamma(q^7\alpha\beta + q^8\alpha^2\beta\delta - (1 + q^4)\beta\delta),$$

etc. Using this form of the generators of \mathcal{Q}_0 one easily finds that $\ker \pi / \mathcal{Q}_0$ is spanned by the projections of the following elements of $\ker \pi$:

$$\alpha^k\beta^{2n-k}, \quad \alpha^k\gamma^{2n-k}, \quad \delta\beta^{2n-1}, \quad \delta\gamma^{2n-1}, \quad n = 1, 2, \dots \quad k = 0, 1, 2, \quad k < 2n.$$

Therefore $\ker \pi / \mathcal{Q}_0$ is an infinite-dimensional vector space.

Notice that for $n = 1$ there are 6 independent elements of $\ker \pi / \mathcal{Q}_0$ coming from monomials in $SO_q(3)$ of degree 1, while for $n > 1$ there are 8 such elements. Using this fact we can compute dimensions of $\ker \pi / \mathcal{Q}^{(k,l)}$. Clearly $\dim(\ker \pi / \mathcal{Q}^{(1,1)}) = 4 = 4(1+1-1)$. Also $\dim(\ker \pi / \mathcal{Q}^{(k,l)}) = \dim(\ker \pi / \mathcal{Q}^{(l,k)})$. First take $k = 1, l > 1$. Then, by counting elements in $\ker \pi / \mathcal{Q}_0$ of given

degree we find $\dim(\ker \pi / \mathcal{Q}^{(1,l)}) = 5 + 4(l - 2) + 3 = 4l = 4(l + 1 - 1)$. Finally take $k, l > 1$. Then $\dim(\ker \pi / \mathcal{Q}^{(k,l)}) = 6 + 4(k - 2) + 4(l - 2) + 6 = 4(k + l - 1)$ as stated.

In the case of $\ker \pi / \mathcal{Q}^{(k,l;r,s)}$ new generators added to $\mathcal{Q}^{(k,l)}$ restrict the dimension by $k - r + l - s$. Therefore $\dim(\ker \pi / \mathcal{Q}^{(k,l;r,s)}) = \dim(\ker \pi / \mathcal{Q}^{(k,l)}) - (k - r + l - s) = 3k + 3l + r + s - 4$. \square

Proposition 4.2 *Let differential structure on $\mathbb{C}[Z, Z^{-1}]$ be given by the ideal \mathcal{Q} generated by $Z^{-1} + q^4 Z - (1 + q^4)$. The largest differential calculus on $SO_q(3)(S_q^2, \mathbb{C}[Z, Z^{-1}], \pi)$ compatible with q -monopole connection is specified by the ideal $\mathcal{Q}_P \subset \ker \epsilon_{SO_q(3)}$ generated by $\beta\gamma$ and $\delta^2 + q^4\alpha^2 - (1 + q^4)$. This calculus is infinite-dimensional. Let $\mathcal{Q}_P^{(k,l;r,s)} = \langle \mathcal{Q}^{(k,l;r,s)}, i(\mathcal{Q})SO_q(3) \rangle$, be a family of right ideals in $\ker \epsilon_{SO_q(3)}$ indexed by $k, l = 1, 2, \dots$, $r = 0, 1, \dots, k$, $s = 0, 1, \dots, l$. Each of $\mathcal{Q}_P^{(k,l;r,s)}$ induces a $3k + 3l + r + s - 3$ -dimensional, left-covariant differential calculus on $SO_q(3)$.*

Proof We need to show that $\mathcal{Q}_P = \langle \mathcal{Q}_0, i(\mathcal{Q})SO_q(3) \rangle$. This is equivalent to showing that the generators of \mathcal{Q}_0 can be expressed as linear combinations of elements of \mathcal{Q}_P . Clearly $\beta\gamma \in \mathcal{Q}_P$. Furthermore we have

$$q^4\alpha^3\beta + \delta\beta - (1 + q^4)\alpha\beta = (\delta^2 + q^4\alpha^2 - (1 + q^4))\alpha\beta - q^{-3}\beta\gamma\delta\beta \in \mathcal{Q}_P,$$

$$q^4\alpha^3\gamma + \delta\gamma - (1 + q^4)\alpha\gamma = (\delta^2 + q^4\alpha^2 - (1 + q^4))\alpha\gamma - q^{-3}\beta\gamma\delta\gamma \in \mathcal{Q}_P.$$

To prove the remaining part of the proposition it suffices to notice that $\ker \epsilon_P / \mathcal{Q}_P$ is spanned by elements of $\ker \epsilon_P / \mathcal{Q}_0$ listed in Proposition 4.1 and additionally by the projection of $\alpha^2 - 1$. Similar calculation as in Proposition 4.1 thus reveals that $\dim(\ker \epsilon_P / \mathcal{Q}_P^{(k,l;r,s)}) = 3k + 3l + r + s - 3$. \square

As a concrete illustration of the above construction we consider differential calculus induced by $\mathcal{Q}_P^{(1,1;1,1)} = \mathcal{Q}_P^{(1,1)}$. Explicitly $\mathcal{Q}_P^{(1,1)}$ is generated by the following four elements $\delta^2 + q^4\alpha^2 - (1 + q^4)$, $\beta^2, \beta\gamma, \gamma^2$. The space $\ker \epsilon / \mathcal{Q}_P^{(1,1)}$ is five-dimensional, so that $\mathcal{Q}_P^{(1,1)}$ generates a five-dimensional left covariant differential calculus $\Omega^1(SO_q(3))$ on $SO_q(3)$. Since $SO_q(3)$ is a subalgebra of $SU_q(2)$ the four elements above generate an ideal in $SU_q(2)$ which also induces a

differential calculus on $SU_q(2)$. Choosing the following basis for the space of left-invariant one forms in $\Omega^1(SO_q(3))$

$$\omega_0 = \frac{1}{q^4 - 1} \pi \mathcal{N} \circ \theta(1 \otimes (q^4 \alpha \beta - \delta \beta)), \quad \omega_2 = -\frac{q^{-1}}{q^4 - 1} \pi \mathcal{N} \circ \theta(1 \otimes (\delta \gamma - q^4 \alpha \gamma)), \quad (8)$$

$$\omega_3 = \frac{1}{q^2 + 1} \pi \mathcal{N} \circ \theta(1 \otimes (\alpha \beta - \delta \beta)), \quad \omega_4 = -\frac{1}{q^2 + 1} \pi \mathcal{N} \circ \theta(1 \otimes (\delta \gamma - \alpha \gamma)), \quad (9)$$

$$\omega_1 = \frac{1}{q^{-2} + 1} \pi \mathcal{N} \circ \theta(1 \otimes (\alpha^2 - 1)), \quad (10)$$

one derives the commutation relations in $\Omega^1(SO_q(3))$ embedded in $\Omega^1(SU_q(2))$,

$$\omega_{0,2} \alpha = q^{-1} \alpha \omega_{0,2}, \quad \omega_{3,4} \alpha = q^{-3} \alpha \omega_{3,4}, \quad \omega_1 \alpha = q^{-2} \alpha \omega_1 + \beta \omega_4, \quad (11)$$

$$\omega_{0,2} \beta = q^1 \beta \omega_{0,2}, \quad \omega_{3,4} \beta = q^3 \beta \omega_{3,4}, \quad \omega_1 \beta = q^2 \beta \omega_1 + \alpha \omega_4, \quad (12)$$

and similarly for α replaced with γ and β replaced with δ . The exact one-forms are given in terms of ω_i as follows

$$d\alpha = \alpha \omega_1 - q\beta(\omega_2 - \frac{q}{1 - q^2} \omega_4), \quad d\beta = -q^2 \beta \omega_1 + \alpha(\omega_0 + \frac{q^2}{1 - q^2} \omega_3),$$

and similarly for α replaced with γ and β replaced with δ . It can be easily checked that the forms $\omega_0, \omega_2, \omega_3, \omega_4$ are horizontal. Note that this calculus reduces to the 3D calculus of Woronowicz if one sets $\omega_3 = \omega_4 = 0$. This is equivalent to enlarging $\mathcal{Q}_P^{(1,1)}$ by $(\delta - \alpha)\beta$, $(\delta - \alpha)\gamma$ and thus the 3D Woronowicz calculus corresponds to $\mathcal{Q}_P^{(1,1;0,0)}$.

The calculus $\mathcal{Q}_P^{(1,1)}$ appears naturally when one looks at the monopole bundle from the local point of view. Recall from [1] that one of the trivialisations of the q -monopole bundle has the form $P_1(M_1, \mathbb{C}[Z, Z^{-1}], \Phi_1)$, where $P_1 = SO_q(3)[(\beta\gamma)^{-1}]$, $M_1 = S_q^2[(b_3 - 1)^{-1}]$, and $\Phi_1(Z^n) = (\beta^{-1}\gamma)^n$, $n \in \mathbb{Z}$. This trivialisation corresponds to the quantum sphere with the north pole removed. It can be easily shown that $P_1 = M_1 \otimes \mathbb{C}[Z, Z^{-1}]$ as an algebra. The structure of M_1 can be most easily described in the stereographic projection coordinates, $z = \alpha\gamma^{-1} = qb_-(b_3 - 1)^{-1}$, $\bar{z} = \delta\beta^{-1} = b_+(b_3 - 1)^{-1}$, introduced in [24]. M_1 is then equivalent to the quantum hyperboloid [25] generated by z, \bar{z} , $(1 - z\bar{z})^{-1}$ and the relation

$$\bar{z}z = q^2 z\bar{z} + 1 - q^2.$$

The natural differential structure $\Omega^1(M_1)$ on M_1 , also discussed in [24], is given by the relations

$$zdz = q^{-2}dz z, \quad zd\bar{z} = q^{-2}d\bar{z} z, \quad \bar{z}dz = q^2dz \bar{z}, \quad \bar{z}d\bar{z} = q^2d\bar{z} \bar{z}.$$

In other words $\Omega^1(M_1) = \Omega^1 M_1 / \mathcal{N}_{M_1}$, where the subbimodule $\mathcal{N}_{M_1} \subset \Omega^1 M_1$ is generated by

$$q^{-2}\bar{z} \otimes z + z \otimes \bar{z} - q^{-2}\bar{z}z \otimes 1 - 1 \otimes z\bar{z}. \quad (13)$$

$$(1 + q^2)z \otimes z - q^2z^2 \otimes 1 - 1 \otimes z^2, \quad (1 + q^{-2})\bar{z} \otimes \bar{z} - q^{-2}\bar{z}^2 \otimes 1 - 1 \otimes \bar{z}^2. \quad (14)$$

The subbimodule M_1 and the q -monopole connection taken as the input data in Proposition 3.3 produce the differential calculus on P_1 which coincides with the differential calculus induced by $\mathcal{Q}_P^{(1,1)}$ when restricted to $SO_q(3)$. Notice also that the generator (13) appears as a consequence of the existence of the minimal horizontal subbimodule \mathcal{N}_0 . Thus the differential structures on P_1 obtained from data $(\mathcal{N}_{M_1}, \omega_D)$ and $(\tilde{\mathcal{N}}_{M_1}, \omega_D)$, where $\tilde{\mathcal{N}}_{M_1}$ is generated by (14) only, are identical.

In any calculus $\Omega^1(SO_q(3))$ admitting the q -monopole connection one can define one-form ω_1 by (10), with $\pi_{\mathcal{N}}$ a canonical projection related to the bimodule \mathcal{N} defining $\Omega^1(SO_q(3))$. Then the connection $\omega_D : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(SO_q(3))$ can be computed explicitly,

$$\omega_D([Z - 1]) = (1 + q^{-2})\omega_1.$$

The canonical map $i_D : \ker \epsilon_{\mathbb{C}[Z, Z^{-1}]} / \mathcal{Q} \rightarrow \ker \epsilon_{SO_q(3)} / \mathcal{Q}_P$, with $\mathcal{N} = \theta(SO_q(3) \otimes \mathcal{Q}_P)$, corresponding to ω_D comes out as $i_D([Z - 1]) = [\alpha^2 - 1]$ and is clearly Ad-covariant since $\text{Ad}([Z - 1]) = [Z - 1] \otimes 1$ and $\overline{\text{Ad}}([\alpha^2 - 1]) = [\alpha^2 - 1] \otimes 1$.

Similarly, regardless of the differential calculus on $P_1(M_1, \mathbb{C}[Z, Z^{-1}], \Phi_1)$, the local connection one-form $\beta : \ker \epsilon_{\mathbb{C}[Z, Z^{-1}]} \rightarrow \Omega^1(M_1)$ can be computed as follows. It is given by

$$\beta(h) = \Phi_1(h_{(1)})Si(h_{(2)})_{(1)}d(i(h_{(2)})_{(2)}\Phi_1^{-1}(h_{(3)})).$$

To compute it explicitly one can use Lemma 3.4 to prove the following

Lemma 4.3 *Let $P(M, \mathbb{C}[Z, Z^{-1}], \Phi)$ be a trivial quantum principal bundle with a trivialisation Φ which is an algebra map. Assume that differential structure $\Omega^1(\mathbb{C}[Z, Z^{-1}])$ is given by the ideal generated by $Z^{-1} + q^4Z - (1 + q^4)$ for q a complex, non-zero parameter. Then $\omega =$*

$\Phi^{-1} * \beta \circ \pi_\epsilon * \Phi + \Phi^{-1} * d \circ \Phi$ is a connection in $P(M, \mathbb{C}[Z, Z^{-1}], \Phi)$ if and only if the map $\beta : \ker \epsilon \rightarrow \Omega^1(M)$ satisfies the following conditions

$$\begin{aligned} \beta(Z^{n+1} - 1) &= (1 + q^{-4})\Phi(Z)\beta(Z^n - 1)\Phi^{-1}(Z) - q^{-4}\Phi(Z^2)\beta(Z^{n-1} - 1)\Phi^{-1}(Z^2) \\ &\quad + (1 + q^{-4})\Phi(Z)d\Phi^{-1}(Z) - q^{-4}\Phi(Z^2)d\Phi^{-1}(Z^2) \end{aligned}$$

$$\begin{aligned} \beta(Z^{-n} - 1) &= (1 + q^4)\Phi^{-1}(Z)\beta(Z^{-n+1} - 1)\Phi(Z) - q^4\Phi^{-1}(Z^2)\beta(Z^{-n+2} - 1)\Phi(Z^2) \\ &\quad + (1 + q^4)\Phi^{-1}(Z)d\Phi(Z) - q^4\Phi^{-1}(Z^2)d\Phi(Z^2), \end{aligned}$$

for any $n \in \mathbb{N}$.

The above lemma implies, in particular, that the map β corresponding to the q -monopole connection is fully determined by its action on $Z - 1$ say, where it is given by

$$\beta(Z - 1) = (1 - z\bar{z})^{-1}(q^2 z d\bar{z} - q^{-2} \bar{z} dz).$$

The above formula for β is valid in any differential structure on M_1 which admits a q -monopole connection, in particular in the natural one discussed above. The map β is related to q -monopole connection ω_D as in Proposition 3.3. The corresponding map $\hat{\Phi}_1$ can be constructed and, applied to the generic element of $\mathcal{N}_{\mathbb{C}[Z, Z^{-1}]}$ of the form $\theta(g \otimes h)$, $g \in \mathbb{C}[Z, Z^{-1}]$, $h \in \mathcal{Q}$, reads $\Phi_1(g)Si(h)_{(1)} \otimes i(h)_{(2)}$.

5 Finite gauge theory and Czech cohomology

In this section we show how quantum differential calculi and gauge theory can be applied in the simplest setting where $M = \mathbb{C}(\Sigma)$, Σ a finite set, and $H = \mathbb{C}(G)$, G a finite group. We consider the case of a tensor product bundle $P = \mathbb{C}(\Sigma) \otimes \mathbb{C}(G)$. We show how this formalism provides a quantum geometrical picture of Czech cohomology when Σ is the indexing set of a good cover of a topological manifold. This demonstrates a possible new direction to the construction of manifold invariants: instead of the usual approach in algebraic topology whereby one looks at the combinatorics of the geometrical structures on manifolds, we consider instead the (quantum) geometry of combinatorial structures on the manifold.

Although we are primarily interested in 1-forms (and occasionally 2-forms), it is important to know that they extend to an entire exterior algebra. Recall that for any unital algebra M there is a universal extension Ω^*M of Ω^1M given in degree n as the joint kernel in $M^{\otimes n+1}$ of all the n maps given by adjacent product. It can be viewed as $\Omega^1M \otimes_M \Omega^1M \otimes_M \cdots \otimes_M \Omega^1M$. The collection Ω^*M forms a differential graded algebra with

$$(a_0 \otimes \cdots \otimes a_n) \cdot (b_0 \otimes \cdots \otimes b_m) = (a_0 \otimes \cdots \otimes a_m b_0 \otimes \cdots \otimes b_m)$$

$$d_U(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n+1} (-1)^j (a_0 \otimes \cdots \otimes a_{j-1} \otimes 1 \otimes a_j \otimes \cdots \otimes a_n)$$

with the obvious conventions for $j = 0, n+1$ understood. A general exterior algebra $\Omega^*(M)$ is then obtained by quotienting it by a differential graded ideal, i.e. an ideal of Ω^*M stable under d_U . Without loss of generality, we always assume that the degree 0 component of the differential ideal is trivial. The degree 1 component is in particular a sub-bimodule \mathcal{N}_M of Ω^1M as in the setting above. Conversely, $\Omega^1(M)$ as defined by a sub-bimodule \mathcal{N}_M has a *maximal prolongation* to an exterior algebra $\Omega^*(M)$ by taking differential ideal generated by $\mathcal{N}_M, d_U \mathcal{N}_M$. In each degree it can be viewed as a quotient of $\Omega^1(M) \otimes_M \Omega^1(M) \otimes_M \cdots \otimes_M \Omega^1(M)$ by the additional relations implied by the Leibniz rule applied to the relations of $\Omega^1(M)$ cf[1]. For example, $\Omega^2(M) = \Omega^1(M) \otimes_M \Omega^1(M) / (\pi_M \otimes_M \pi_M)(d_U \mathcal{N}_M)$, where π_M is the canonical projection $\Omega^1M \rightarrow \Omega^1(M)$.

Clearly one may take a similar view for $\Omega^2(M)$. The degree 2 part of a differential ideal of Ω^*M will, in particular, be a subbimodule \mathcal{F} in the range

$$\overline{\mathcal{N}_M} \subseteq \mathcal{F} \subseteq \Omega^2M$$

where $\overline{\mathcal{N}_M} = (\Omega^1M)\mathcal{N}_M + \mathcal{N}_M(\Omega^1M) + d_U \mathcal{N}_M$ is a subbimodule (in view of the Leibniz rule for d_U), and $\Omega^2(M) = \Omega^2M / \mathcal{F}$. Conversely, given $\Omega^1(M)$, any subbimodule \mathcal{F} in this range defines an $\Omega^2(M)$ compatible with $\Omega^1(M)$ in the natural way. Moreover, taking the differential ideal generated by $\mathcal{N}_M, \mathcal{F}, d_U \mathcal{F}$ provides a prolongation of $\Omega^1(M), \Omega^2(M)$ as specified by $\mathcal{N}_M, \mathcal{F}$. Similarly, one may specify the exterior algebra up to any finite degree and know that it prolongs to an entire exterior algebra $\Omega^*(M)$. This is the point of view which we take throughout the paper.

We begin with a lemma which is well-known (see e.g. [26, p. 184]), but which we include because it provides the framework for our analysis of Ω^1 and Ω^2 in the case of a discrete set.

Lemma 5.1 *When Σ is a finite set of order $|\Sigma|$, $\Omega^n \mathbb{C}(\Sigma)$ may be identified with the subset $\mathbb{C}^{|\Sigma|} \otimes \cdots \otimes \mathbb{C}^{|\Sigma|}$ consisting of degree- $(n+1)$ tensors vanishing on any adjacent diagonal. The exterior derivative $\Omega^{n-1} \mathbb{C}(\Sigma) \rightarrow \Omega^n \mathbb{C}(\Sigma)$ is*

$$(\mathrm{d}_U f)_{i_0, \dots, i_n} = \sum_{j=0}^{n+1} (-1)^j f_{i_0, \dots, \hat{i}_j, \dots, i_n}$$

where $\hat{}$ denotes omission. The algebra structure of $\Omega \mathbb{C}(\Sigma)$ is $(f \cdot g)_{i_0 \dots i_{n+m}} = f_{i_0 \dots i_n} g_{i_{n+1} \dots i_{n+m}}$ for f of degree n and g of degree m .

Proof We consider $\mathbb{C}(\Sigma)$ as a vector space with basis Σ . An element is then a vector in \mathbb{C}^n with components f_i for $i \in \Sigma$. The corresponding function is $f = \sum_i f_i \delta_i$ where δ_i is the Kronecker delta-function at i . We have $\Omega^n \mathbb{C}(\Sigma)$ as a subspace of $\mathbb{C}(\Sigma)^{\otimes n+1}$ in the kernel of adjacent product maps. These send $\sum f_{i_0 \dots i_n} \delta_{i_0} \otimes \cdots \delta_{i_n}$ to $\sum f_{i_0 \dots i_{j-1}, i_{j+1} \dots i_n} \delta_{i_0} \otimes \cdots \otimes \delta_{i_n}$ for all $j = 1$ to $j = n$. So the joint kernel means tensors f_{i_0, \dots, i_n} vanishing on the identification of any two adjacent indices. The action of d_U on $\Omega^{n-1} \mathbb{C}(\Sigma)$ is a signed insertion of 1 in each position of the n -fold tensor product, which is the form stated. The product structure is the pointwise product with the outer copies of $\mathbb{C}(\Sigma)$, as stated. \square

In particular, we identify $\Omega^1 \mathbb{C}(\Sigma)$ with $|\Sigma| \times |\Sigma|$ matrices vanishing on the diagonal.

Proposition 5.2 *Let Σ be a finite set. Then the possible $\Omega^1(\mathbb{C}(\Sigma))$ are in 1-1 correspondence with subsets $E \subset \Sigma \times \Sigma - \text{Diag}$. The quotient $\Omega^1(\mathbb{C}(\Sigma))$ is obtained by setting to zero the matrix entries f_{ij} for which $(i, j) \notin E$. In this way we identify $\Omega^1(\mathbb{C}(\Sigma)) = \mathbb{C}(E)$.*

Proof We consider first the possible sub-bimodules $\mathcal{N}_M \subset \Omega^1 \mathbb{C}(\Sigma)$. Let δ_i denote the obvious (Kronecker delta-function) basis elements of $\mathbb{C}(\Sigma)$. If $\lambda \delta_i \otimes \delta_j + \mu \delta_{i'} \otimes \delta_{j'} \in \mathcal{N}_M$ for $(i, j) \neq (i', j')$ then multiplying by δ_i from the left or by δ_j from the right implies that $\lambda \delta_i \otimes \delta_j \in \mathcal{N}_M$ also, as \mathcal{N}_M is required to be a sub-bimodule. Hence $\mathcal{N}_M = \text{span}\{\delta_i \otimes \delta_j\}$ for (i, j) in some subset of $\Sigma \times \Sigma - \text{Diagonal}$. We denote the complement of this subset in $\Sigma \times \Sigma - \text{diag}$ by E . This gives the general form of a nonuniversal $\Omega^1(\mathbb{C}(\Sigma)) = \Omega^1 \mathbb{C}(\Sigma) / \mathcal{N}_M$. \square

We write $i - j$ whenever $(i, j) \in E$ and we write $i \# j$ whenever (i, j) is in the complement of E in $\Sigma \times \Sigma - \text{diag}$.

Lemma 5.3 *Let $\Omega^1(\mathbb{C}(\Sigma))$ be defined as above by E . Then the possible $\Omega^2(\mathbb{C}(\Sigma))$ extending this are in 1-1 correspondence with vector subspaces $V_{ij} \subset \mathbb{C}(\Sigma - \{i, j\})$ such that*

$$V_{ik} \ni \begin{cases} \sum_{j \neq i, k} \delta_j & \text{if } i \# k \\ \delta_j & \text{if } i \# j, j \neq k \\ \delta_j & \text{if } i \neq j, j \# k \end{cases}.$$

Then $\Omega^2(\mathbb{C}(\Sigma)) = \oplus_{i, k} \delta_i \otimes \mathbb{C}(\Sigma - \{i, k\}) / V_{ik} \otimes \delta_k$. We say that $\Omega^2(\mathbb{C}(\Sigma))$ is local if all the V_{ik} are spanned by δ -function basis elements.

Proof We first compute $\overline{\mathcal{N}_M}$. Clearly, $\mathcal{N}_M(\Omega^1 \mathbb{C}(\Sigma)) = \text{span}\{\delta_i \otimes \delta_j \otimes \delta_k \mid \forall i \# j, k \neq j\}$ and $(\Omega^1 \mathbb{C}(\Sigma))\mathcal{N}_M = \text{span}\{\delta_i \otimes \delta_j \otimes \delta_k \mid \forall i \neq j, k \# j\}$, while for $i \# j$, $d_U \delta_i \otimes \delta_j = 1 \otimes \delta_i \otimes \delta_j - \delta_i \otimes 1 \otimes \delta_j + \delta_i \otimes \delta_j \otimes 1$ has most of its terms contained already in the above. The additional contribution to $\overline{\mathcal{N}_M}$ is $\{\delta_i \otimes (\sum_{a \neq i, j} \delta_a) \otimes \delta_j \mid i \# j\}$. These three subspaces span $\overline{\mathcal{N}_M}$. Meanwhile, by similar arguments to the proof of Lemma 5.1, any $\mathbb{C}(\Sigma)$ -bimodule $\mathcal{F} \subset \Omega^2 \mathbb{C}(\Sigma)$ has the form

$$\mathcal{F} = \text{span}\{\delta_i \otimes V_{ik} \otimes \delta_k \mid i, k \in \Sigma\}, \quad V_{ik} \subseteq \mathbb{C}(\Sigma - \{i, k\})$$

for some vector subspaces as shown. In order to contain $\overline{\mathcal{N}_M}$ we see that we require the subspaces V_{ik} to contain the elements stated. \square

The local case is clearly the natural one for ‘geometry’ on the set Σ . From Proposition 5.2 we know that $\Omega^1(\mathbb{C}(\Sigma))$ is always local in the same sense. From the above lemma we see that its maximal prolongation has the $V_{ij} = 0$ except in the cases stated, when it is spanned by the stated vectors; it is therefore not local and we need to quotient it further.

Theorem 5.4 *Local $\Omega^2(\mathbb{C}(\Sigma))$ are in correspondence with subsets*

$$F_0 \subseteq \{(i, j) \in \Sigma \times \Sigma \mid i - j, j - i\}, \quad F \subseteq \{(i, j, k) \in \Sigma \times \Sigma \times \Sigma \mid i - j, j - k, i - k\}.$$

Then $\Omega^2(\mathbb{C}(\Sigma)) = \mathbb{C}(F) \oplus \mathbb{C}(F_0)$ can be identified with 3-tensors f_{ijk} vanishing on adjacent diagonals and such that either $i = k, (i, j) \in F_0$ or $(i, j, k) \in F$.

The 1-cycles in $\Omega^1(\mathbb{C}(\Sigma))$ are f_{ij} such that

$$f_{ij} = -f_{ji}, \quad f_{ij} - f_{ik} + f_{jk} = 0$$

for all $(i, j) \in F_0$ and $(i, j, k) \in F$ respectively. Moreover, the image of $\mathbb{C}(\Sigma)$ is $(dg)_{ij} = g_i - g_j$ for all $i - j$.

Proof In the preceding lemma we consider V_{ik} as spanned by δ -functions on the complement of some subsets $F_{ik} \subseteq \Sigma - \{i, k\}$, say. We consider the requirements of the lemma for the three mutually exclusive possible cases $i \# k$, $i = k$ and $i - k$. To contain $\sum_{j \neq i, k} \delta_j$ in the first case, we need $F_{ik} = \emptyset$. For the second case, we know that $i \# j$ or $j \# i$ must imply j not in F_{ii} , i.e. $j \in F_{ii}$ should imply $i - j$ and $j - i$ (we consider only $j \in \Sigma - \{i\}$). This requires $F_{ii} \subset \{j | i - j, j - i\}$. Similarly for the third possibility. Thus, the conditions on V_{ik} in the preceding lemma become now

$$F_{ik} \subseteq \begin{cases} \emptyset & \text{if } i \# k \\ \{j \in \Sigma | i - j, j - i\} & \text{if } i = k \\ \{j \in \Sigma | i - j, j - k\} & \text{if } i - k \end{cases}.$$

Moreover, in the local case we can identify the quotients as remaining basis elements, i.e. $\Omega^2(\mathbb{C}(\Sigma)) = \oplus_{i, k} \delta_i \otimes \mathbb{C}(F_{ik}) \otimes \delta_k$.

Next, we can collect together all the F_{ik} where $i - k$. The specification of these is equivalent to the specification of F as stated. Likewise, the specification of all the F_{ii} is equivalent to the specification of F_0 as stated. Then $\Omega^2(\mathbb{C}(\Sigma)) = \mathbb{C}(F) \oplus \mathbb{C}(F_0)$ where $\mathbb{C}(F)$ refers to the coefficients of vectors of the form $\delta_i \otimes \delta_j \otimes \delta_k$ when $i - j, j - k, i - k$, and $\mathbb{C}(F_0)$ refers to coefficients of $\delta_i \otimes \delta_j \otimes \delta_i$.

Finally, we compute the $(df)_{iji} = f_{ij} + f_{ji}$ and $(df)_{ijk} = f_{ij} - f_{ik} + f_{jk}$ in $\Omega^2(\mathbb{C}(\Sigma))$, where we need only consider $(i, j) \in F_0$ in the first equation and $(i, j, k) \in F$ in the second. Hence the closed forms are as stated. \square

It should be clear that a similar situation occurs to all orders. The maximal prolongation of local Ω^1, Ω^2 , say, will not be local, requiring further subset data to obtain local Ω^3 , and so on. Note also that such ‘finite differential geometry’ makes no sense classically because 1-forms and functions commute in $\Omega^1(\mathbb{C}(\Sigma))$ only in the trivial case; one needs the more general axioms of quantum differential geometry and quantum exterior algebra. As an application,

we may associate a suitable nonuniversal quantum differential calculus to any finite cover of a topological manifold, i.e. we have the possibility to do ‘geometry’ on the combinatorics of the manifold rather than combinatorics of the geometry. We recall that a finite cover $\{U_i\}$ has some nonzero intersections $\{U_i \cap U_j\}$, some nonzero triple intersections $\{U_i \cap U_j \cap U_k\}$ etc.

Corollary 5.5 *Let X be a topological manifold with a finite good open cover $\{U_i\}$ where i run over an indexing set Σ . The cover has an associated local quantum differential calculus $\Omega^1(\mathbb{C}(\Sigma))$, $\Omega^2(\mathbb{C}(\Sigma))$ such that its quantum cohomology is the Czech cohomology $H^1(X)$.*

Proof Let E be the distinct pairs for which $U_i \cap U_j \neq \emptyset$. Here $i - j$ iff $j - i$ so E has a symmetric form. We take $F_0 = E$. We take for F the distinct triples for which $U_i \cap U_j \cap U_k \neq \emptyset$. We have 1-cochains $\{f_{ij}\}$ defined for $i - j$ but we do not require $f_{ji} = -f_{ij}$ for the cochain itself, i.e there are many more 1-cochains than in Czech cohomology. On the other hand, the closure condition is stronger than in Czech cohomology and antisymmetry appears ‘on shell’ for any closed cochain. The image of d in $\Omega^1(\mathbb{C}(\Sigma))$ has the usual (antisymmetric) form, so we recover the usual $H^1(X)$ in spite of the ‘quantum’ construction. \square

Note that for a smooth compact manifold this recovers the DeRahm cohomology $H^1(X)$, i.e. we recover a known geometrical invariant from ‘geometry’ directly on the cover. Also, it should be clear that the similar result applies more generally to any simplicial complex (with the one in the corollary being the nerve of the cover of a topological manifold.) We let Σ be the vertices, E the edges and F the faces. The associated quantum exterior algebra $\Omega^*(\mathbb{C}(\Sigma))$ is such that its cohomology H^1 coincides with the usual simplicial cohomology. Unlike the usual situation, however, our ‘quantum’ resolution of the simplicial cohomology has the cochains forming a differential graded algebra and not only a complex of vector spaces as in the usual situation. This allows us to proceed in a ‘geometrical’ fashion. Essentially, the product in Lemma 5.1 is not compatible with antisymmetry of the cochains and we instead impose the antisymmetry only ‘on shell’ and not for the cochains themselves. Although the similarity of d_U in Lemma 5.1 with the Czech coboundary is obvious from the outset, one usually imposes antisymmetry by hand on the cochains (see for example [27]) and hence loses the exterior algebra structure.

We may now proceed to consider further geometrical structures in this discrete setting. In particular, gauge theory or quantum group gauge theory then provides the natural extension to group or quantum-group valued Czech cohomology. We note first that if we are interested in only trivial principal bundles and gauge theory in terms of the base M , we do not need to fix a differential calculus $\Omega^1(H)$. We need only the coalgebra structure of $H[1]$ for a formal gauge theory with any $\beta : H \rightarrow \Omega^1(M)$ (not necessarily vanishing on 1) and any $\gamma : H \rightarrow M$ (not necessarily unital). As explained in [28] we can use any nonuniversal $\Omega^1(M), \Omega^2(M)$ which are compatible (as part of a differential graded algebra), and still have the fundamental lemma of gauge theory that

$$F(\beta) = d\beta + \beta * \beta; \quad \beta^\gamma = \gamma^{-1} * \beta * \gamma + \gamma^{-1} * d\gamma$$

obeys

$$F(\beta^\gamma) = \gamma^{-1} * F(\beta) * \gamma$$

where $*$ denotes the convolution product defined via the coproduct of H . We can still have sections and covariant derivatives as well at this level[28]. Equally well, we can work with $\beta \in \Omega^1(M) \otimes A$ and invertible $\gamma \in M \otimes A$, where A need only be a unital algebra. For example, the zero curvature equation $d\beta + \beta * \beta = 0$ makes sense in $\Omega^2(M) \otimes A$.

Proposition 5.6 *Let A be a unital algebra and consider gauge fields $\beta \in \Omega^1(\mathbb{C}(\Sigma)) \otimes A$ such that $F(\beta) = 0$ in $\Omega^2(\mathbb{C}(\Sigma)) \otimes A$. There is an action of the group of invertible elements $\gamma \in \mathbb{C}(\Sigma) \otimes A$ on this space and the moduli space of zero curvature gauge fields modulo such transformations coincides with the multiplicative Czech cohomology $H^1(X, A)$ in the setting of the preceding corollary.*

Proof In the setting of Proposition 5.4 we have

$$F(\beta)_{iji} = \beta_{ij} + \beta_{ji} + \beta_{ij}\beta_{ji}, \quad F(\beta)_{ijk} = \beta_{ij} + \beta_{jk} - \beta_{ik} + \beta_{ij}\beta_{jk}$$

for all $(i, j) \in F_0$ and $(i, j, k) \in F$ respectively. Hence the zero-curvature equation is

$$(1 + \beta_{ij})(1 + \beta_{ji}) = 1, \quad (1 + \beta_{ij})(1 + \beta_{jk}) = 1 + \beta_{ik}$$

as a multiplicative version of Proposition 5.4 and with values in A . Although β_{ij} are not imposed to be such that $g_{ij} = 1 + \beta_{ij}$ is invertible, we see that this appears ‘on shell’ for zero curvature gauge fields, along with $g_{ij}^{-1} = g_{ji}$. Finally, a gauge transformation means $\gamma \in \mathbb{C}(\Sigma) \otimes A$ with components $\{\gamma_i\}$ invertible, and the action on connections is

$$\beta_{ij}^\gamma = \gamma_i^{-1} \beta_{ij} \gamma_j + \gamma_i^{-1} \gamma_j - 1$$

for all $i - j$. Hence, in the particular setting of Corollary 5.5 (or more generally for a simplicial complex) we obtain for the moduli space of zero curvature gauge fields the multiplicative Czech cohomology. \square

Note that if A supports logarithms then $1 + \beta_{ij} = \exp f_{ij}$ and the multiplicative theory becomes equivalent to the additive theory as in Corollary 5.5, i.e. we have a second interpretation with f as A -valued quantum differential forms in this case.

We proceed now to quantum group gauge theory with a full quantum geometric structure where $P = \mathbb{C}(\Sigma) \otimes \mathbb{C}(G) = \mathbb{C}(\Sigma \times G)$, G a finite group (say) and both $\mathbb{C}(G)$, $\mathbb{C}(\Sigma)$ are equipped with quantum differential calculi. Bicovariant (coirreducible) calculi on $\mathbb{C}(G)$ are known to correspond to nontrivial conjugacy classes on G . When $G = \mathbb{Z}_2$ there is only one non-zero calculus, which is also the universal one. Here $\ker \epsilon$ is 1-dimensional so β, γ are fully specified as $\beta \in \Omega^1(\mathbb{C}(\Sigma))$ and $\gamma \in \mathbb{C}(\Sigma)$ with invertible components. In this case we recover the setting of Proposition 5.6 with $A = \mathbb{C}$. However, for other groups (or if we use the zero calculus on $\mathbb{C}(\mathbb{Z}_2)$) we need the theory of quantum principal bundles with nonuniversal calculi developed in Section 3. We demonstrate some of this theory now, namely Proposition 3.3 which provides the construction of the differential calculus on a trivial bundle P by ‘gluing’ the chosen calculi on the base and on the fibre via a universal connection.

We consider $G = \mathbb{Z}_3 = \{e, g, g^2\}$, which has two non-zero bicovariant calculi, associated to g or g^{-1} . Without loss of generality we consider the one associated to g . Then $\Omega^1(\mathbb{C}(\mathbb{Z}_3))$ is 1-dimensional over $H = \mathbb{C}(\mathbb{Z}_3)$. The unique normalised left-invariant 1-form is ω_1 say and

$$d\delta_e = (\delta_{g^2} - \delta_e)\omega_1, \quad d\delta_g = (\delta_e - \delta_g)\omega_1, \quad \omega_1 \delta_{g^i} = \delta_{g^{i-1}}\omega_1$$

gives its structure on a δ -function basis of $\mathbb{C}(\mathbb{Z}_3)$. The ideal \mathcal{Q} for this bicovariant calculus is

$\mathcal{Q} = \mathbb{C}\delta_{g^2}$. From the point of view of Proposition 5.2, the calculus corresponds to edges specified by $a - b$ iff $a = b - 1$, where $a, b \in \{0, 1, 2\} \bmod 3$. The corresponding subbimodule of $\Omega^1\mathbb{C}(\mathbb{Z}_3)$ is $\text{span}\{\delta_e \otimes \delta_{g^2}, \delta_g \otimes \delta_e, \delta_{g^2} \otimes \delta_g\}$.

Example 5.7 Let $\mathbb{C}(\Sigma)$ have differential calculus described by a collection of edges $\{i - j\}$ via Proposition 5.2. Let $\mathbb{C}(\mathbb{Z}_3)$ have the standard 1-dimensional calculus as above. For any $\beta_U : \ker \epsilon \rightarrow \Omega^1\mathbb{C}(\Sigma)$, i.e. a pair $\beta^{(1)} = \beta_U(\delta_g)$, $\beta^{(2)} = \beta_U(\delta_{g^2})$ of $|\Sigma| \times |\Sigma|$ of matrices with zero diagonal, the induced $\Omega^1(\mathbb{C}(\Sigma \times \mathbb{Z}_3))$ via Proposition 3.3 has the allowed edges

$$\begin{aligned} (i, a) - (j, a) & \quad \text{if } i - j, \beta_{ij}^{(2)} = 0 \\ (i, a) - (i, b) & \quad \text{if } a = b - 1 \\ (i, a - 1) - (j, a) & \quad \text{if } i - j, \beta_{ij}^{(1)} = 0 \end{aligned}$$

Moreover, $\omega : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(\mathbb{C}(\Sigma \times \mathbb{Z}_3))$ defined by

$$\begin{aligned} \omega(\delta_g) = & \sum_{i-j, \beta_{ij}^{(2)}=0} \sum_a \beta_{ij}^{(1)} \delta_i \otimes \delta_{g^a} \otimes \delta_j \otimes \delta_{g^a} - \sum_{i-j, \beta_{ij}^{(1)}=0} \sum_a \delta_i \otimes \delta_{g^{a-1}} \otimes \delta_j \otimes \delta_{g^a} \\ & - \sum_{i,a} \delta_i \otimes \delta_{g^{a-1}} \otimes \delta_i \otimes \delta_{g^a} \end{aligned}$$

is a connection on $\mathbb{C}(\Sigma \times \mathbb{Z}_3)$ as a quantum principal bundle with this quantum differential calculus.

Proof Since $P = \mathbb{C}(\Sigma) \otimes \mathbb{C}(\mathbb{Z}_3)$ is a tensor product bundle $P = M \otimes H$, the trivialisation in Proposition 3.3 is $\Phi(h) = 1 \otimes h$ and so

$$\omega_U(h) = (1 \otimes Sh_{(1)})\beta_U(\pi_\epsilon(h_{(2)})) \otimes h_{(3)} + 1 \otimes Sh_{(1)} \otimes 1 \otimes h_{(2)} - 1 \otimes 1 \otimes 1 \otimes \epsilon(h).$$

To compute the minimal horizontal subbimodule

$$\mathcal{N}_0 = P\text{span}\{(m \otimes h_{(1)})\omega_U(qh_{(2)}) - \omega_U(q)(m \otimes h) \mid q \in \mathcal{Q}, m \in M, h \in H\}$$

and $\mathcal{N} = \langle P\mathcal{N}_0P, P\omega_U(\mathcal{Q})P \rangle$ defining $\Omega^1(P)$, we compute first

$$\begin{aligned} \omega_U(\delta_{g^2}) = & \sum_{a+b+c=2} \delta_{g^{-a}} \beta_U(\delta_{g^b}) \delta_{g^c} + \sum_{a+b=2} 1 \otimes \delta_{g^{-a}} \otimes 1 \otimes \delta_{g^b} \\ = & \sum_{i,j,a} \beta_{ij}^{(1)} \delta_i \otimes \delta_{g^{a-1}} \otimes \delta_j \otimes \delta_{g^a} + \sum_{i,j,a} \beta_{ij}^{(2)} \delta_i \otimes \delta_{g^a} \otimes \delta_j \otimes \delta_{g^a} - \sum_a 1 \otimes \delta_{g^{a+1}} \otimes 1 \otimes \delta_{g^a} \end{aligned}$$

where indices a, b, c are taken in $\{0, 1, 2\} \bmod 3$ and $i, j \in \Sigma$. Then

$$\begin{aligned}
& (\delta_l \otimes \delta_{g^b}) \omega_U(\delta_{g^2})(\delta_k \otimes \delta_{g^a}) \\
&= (\delta_l \otimes \delta_{g^b} \otimes 1 \otimes 1) \times \\
& \times \left(\sum_j \beta_{jk}^{(1)} \otimes \delta_j \otimes \delta_{g^{a-1}} \otimes \delta_k \otimes \delta_{g^a} + \sum_j \beta_{jk}^{(2)} \delta_j \otimes \delta_{g^a} \otimes \delta_k \otimes \delta_{g^a} - 1 \otimes \delta_{g^{a+1}} \otimes \delta_k \otimes \delta_{g^a} \right) \\
&= \delta_{b,a-1} \beta_{lk}^{(1)} \delta_l \otimes \delta_{g^{a-1}} \otimes \delta_k \otimes \delta_{g^a} + \delta_{b,a} \beta_{lk}^{(2)} \delta_l \otimes \delta_{g^a} \otimes \delta_k \otimes \delta_{g^a} + \delta_{b,a+1} \delta_l \otimes \delta_{g^{a+1}} \otimes \delta_k \otimes \delta_{g^a}
\end{aligned}$$

Choosing $b = a - 1, a, a + 1$ we see that

$$\begin{aligned}
P\omega_U(\mathcal{Q})P &= \text{span}\{\delta_i \otimes \delta_{g^{a-1}} \otimes \delta_j \otimes \delta_{g^a} | \beta_{ij}^{(1)} \neq 0\} + \text{span}\{\delta_i \otimes \delta_{g^a} \otimes \delta_j \otimes \delta_{g^a} | \beta_{ij}^{(2)} \neq 0\} \\
&+ \text{span}\{\delta_i \otimes \delta_{g^{a+1}} \otimes \delta_j \otimes \delta_{g^a}\}.
\end{aligned}$$

This and

$$P\mathcal{N}_M P = \text{span}\{\delta_i \otimes \delta_{g^a} \otimes \delta_j \otimes \delta_{g^b} | i \neq j\}$$

gives \mathcal{N} . One may compute \mathcal{N}_0 similarly, noting that since $\mathcal{Q} = \mathbb{C}\delta_{g^2}$,

$$\mathcal{N}_0 = \text{span}\{(\delta_l \otimes \delta_{g^b} \otimes 1 \otimes 1) \left((\delta_k \otimes \delta_{g^{a+1}}) \omega_U(\delta_{g^2}) - \omega_U(\delta_{g^2})(\delta_k \otimes \delta_{g^a}) \right)\}.$$

This turns out to be the $P\omega_U(\mathcal{Q})P$ in which its third part is restricted to $\text{span}\{\delta_i \otimes \delta_{g^{a+1}} \otimes \delta_j \otimes \delta_{g^a} | i \neq j\}$.

Next, we compute the edges corresponding to \mathcal{N} as in the setting of Proposition 5.2. We consider only $(i, a) \neq (j, b)$. Then $(i, a) \# (j, b)$ whenever $a = b + 1$ or $(a = b - 1, \beta_{ij}^{(1)} \neq 0)$ or $(a = b, \beta_{ij}^{(2)} \neq 0)$. So the complementary set is $(i, a) - (j, b)$ whenever $(a = b \text{ or } a = b - 1) \text{ and } (i = j \text{ or } i - j) \text{ and } (a = b \text{ or } \beta_{ij}^{(1)} = 0) \text{ and } (a = b - 1 \text{ or } \beta_{ij}^{(2)} = 0)$, which simplifies as stated.

Finally, Proposition 3.3 also provides for a connection $\omega : \ker \epsilon / \mathcal{Q} \rightarrow \Omega^1(P)$. In our case we identify $\ker \epsilon / \mathcal{Q} = \mathbb{C}\delta_g$. Then

$$\omega_U(\delta_g) = \sum_{i,j,a} \beta_{ij}^{(1)} \delta_i \otimes \delta_{g^a} \otimes \delta_j \otimes \delta_{g^a} + \sum_{i,j,a} \beta_{ij}^{(2)} \delta_i \otimes \delta_{g^{a+1}} \otimes \delta_j \otimes \delta_{g^a} - \sum_a 1 \otimes \delta_{g^{a-1}} \otimes 1 \otimes \delta_{g^a}.$$

We then project this down by setting to zero elements in \mathcal{N} , which gives the result as shown. In specific examples one may also compute $\Omega_P^1(M)$ obtained by restricting $\Omega^1(P)$ to M (in general it will not be our original $\Omega^1(M)$, having instead the new subbimodule $\Omega^1 M \cap \mathcal{N}$). \square

We see that a connection β_U ‘glues’ the differential calculus in $\mathbb{C}(G)$ to that on $\mathbb{C}(\Sigma)$ to obtain a differential calculus on the total space. We can of course take quantum groups other than $\mathbb{C}(G)$. For example, we may take $H = \mathbb{C}G$, G a finite group. When G is non Abelian, H is not the function algebra on any space, so this is a genuine application of ‘noncommutative geometry’. In this case we know from [13] that (coirreducible) bicovariant calculi $\Omega^1(\mathbb{C}G)$ may be identified with pairs (V, λ) where V is an (irreducible) representation and $\lambda \in P(V^*)$. We will construct nonuniversal calculi and connections on bicrossproduct bundles of this type (i.e. with fibre $\mathbb{C}G$) in the next section.

One can (in principle) consider other connections on this bundle, the zero curvature condition etc., and obtain in this way (in view of Proposition 5.6) a slew of refinements of Czech cohomology with values in quantum groups equipped with quantum differential structures. Recall that at the level of naive gauge theory as in Proposition 5.6 only the coalgebra of H enters. Thus $H = \mathbb{C}G$ just yields $|G| - 1$ copies of the 1-dimensional gauge theory. By contrast, the theory with nonuniversal calculi on the fibre and bundle carries much more information, including the group structure and (in the case of $\mathbb{C}G$) the choice of (V, λ) . One also has extensions of the geometric theory of quantum principal bundles where the fibre is a braided group or only a coalgebra[11][29]. In a dual form it means gauge fields with values in algebras (not necessarily Hopf algebras) equipped with differential calculi.

Finally, the extension of these ideas to values in a sheaf is also important. Valuation of the usual Czech H^1 in the structure sheaf provides of course a classification of line bundles over X , etc. By taking more exotic Hopf algebras and differential calculi in a sheaf setting we may obtain more interesting invariants and ‘quantum geometrical’ methods to compute them. A further long-range suggestion provided by the above result is that the role of an ‘open cover’ can be naturally encoded as a discrete algebra (here $\mathbb{C}(\Sigma)$) and the choice of nonuniversal differential calculus on it. One may be able to turn this around and take a discrete algebra M and choice of $\Omega(M)$ on it as the starting point for the definition of a quantum manifold ‘with cover M ’. One should then define a ‘sheaf over $M, \Omega(M)$ ’, etc. These are directions to be explored elsewhere.

6 Bundles and Connections on Cross Product Hopf Algebras

As noted already in Section 2, a general trivial quantum principal bundle has the form of a cocycle cross product. Here we will consider in detail some special cases of such cross products where the total space P is itself a Hopf algebra. This covers many of the Hopf algebras in the literature, providing for them natural calculi and connections. This is a further concrete application of quantum group gauge theory and provides a uniform approach to the different kinds of cross product.

In fact, there are mainly two different general constructions for Hopf algebras where the algebra part is a cross product. The first, the bicrossproduct construction[4] associates quantum groups to group factorisations. The other is a bosonisation construction[30] which provides the Borel and maximal parabolic parts of the quantum groups $U_q(g)$, as well as a way of thinking about the quantum double[5][7] and Poincaré quantum groups[6]. Slightly more general is a biproduct construction[31][5], with the starting point being a braided group.

Note that if a homogeneous bundle as in Example 2.3 is split by a coalgebra map $i : H \rightarrow P$ then (a) the bundle is trivial by $\Phi = i$ and (b) the Ad-invariance condition in Example 2.3 holds and the canonical connection $Si(h)_{(1)}di(h)_{(2)}$ coincides with the trivial $\beta = 0$ connection in Example 3.3. The bosonisations are of this type (in fact, i a Hopf algebra map), while bicrossproducts are not in general of this type, although the bundle is still trivial.

6.1 Bicrossproducts

We recall [10] that a general extension of Hopf algebras has the form of a bicrossproduct

$$M \rightarrow M \blacktriangleright H \rightarrow H$$

possibly with cocycles.

We consider the cocycle-free case. In this case H acts on M and M coacts on H and the Hopf algebra structure is the associated cross product and cross coproduct (or ‘bicrossproduct’) from [4]. In this case it is immediate to see from the explicit formulae that $\pi : M \blacktriangleright H \rightarrow H$, $\pi(m \otimes h) = \epsilon(m)h$ is a homogeneous quantum principal bundle over M . Moreover, the map $\Phi : H \rightarrow M \blacktriangleright H$, $\Phi(h) = 1 \otimes h$ is an algebra map. It is easy to see that $\Delta_R \circ \Phi = (\Phi \otimes \text{id}) \circ \Delta$

and $\Phi^{-1} = \Phi \circ S$, so that $M \blacktriangleright H$ as a bundle is trivial. From Proposition 3.3 we already know that natural calculi $\Omega^1(P)$ are provided by the choice of connection defined by $\beta_U : \ker \epsilon \rightarrow \Omega^1 M$. We provide now a construction for suitable β_U such that the resulting $\Omega^1(P)$ is left-invariant.

Proposition 6.1 *Strong, left-invariant connections in $M \blacktriangleright H$ as a trivial quantum principal bundle are in 1-1 correspondence with linear left-invariant maps $\beta_U : \ker \epsilon \rightarrow \Omega^1 M$ such that*

$$\beta_U(\pi_\epsilon(h_{(1)}))h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} - h_{(1)}^{(\bar{2})}\beta_U(\pi_\epsilon(h_{(2)})) \otimes h_{(1)}^{(\bar{1})} = d_U h^{(\bar{2})} \otimes h^{(\bar{1})}$$

Moreover, such β_U are in 1-1 correspondence with linear maps $\gamma : H \rightarrow M$ obeying $\gamma(1) = 1$ and $\epsilon_M \circ \gamma = \epsilon_H$, and such that

$$\gamma(h_{(1)})h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} = \gamma(h_{(2)}) \otimes h_{(1)}, \quad \forall h \in H.$$

The correspondence is via

$$\beta_U(h) = (S\gamma(h)_{(1)})d_U\gamma(h)_{(2)}.$$

The corresponding ω_U is a canonical connection for a splitting map $i(h) = \gamma(h_{(1)}) \otimes h_{(2)}$.

Proof First recall some basic facts about bicrossproducts that are needed for the proof. The definition of a coproduct in $M \blacktriangleright H$, $\Delta(m \otimes h) = m_{(1)} \otimes h_{(1)}^{(\bar{1})} \otimes m_{(2)}h_{(1)}^{(\bar{2})} \otimes h_{(2)}$ implies that $\Delta\Phi(h) = \Phi(h_{(1)}^{(\bar{1})}) \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})$. Let $\alpha : H \rightarrow M \otimes H$ denote a right coaction of M on H used for the definition of $M \blacktriangleright H$, i.e. $\alpha(h) = h^{(\bar{1})} \otimes h^{(\bar{2})}$. Then the property $\alpha(gh) = g_{(1)}^{(\bar{1})}h^{(\bar{1})} \otimes g_{(1)}^{(\bar{2})}(g_{(2)} \triangleright h^{(\bar{2})})$ implies

$$\begin{aligned} 1_H \otimes \Phi(Sh) &= \alpha(Sh_{(2)}h_{(3)})\Phi(Sh_{(1)}) = Sh_{(3)}^{(\bar{1})}h_{(4)}^{(\bar{1})} \otimes Sh_{(3)}^{(\bar{2})}(Sh_{(2)} \triangleright h_{(4)}^{(\bar{2})})\Phi(Sh_{(1)}) \\ &= Sh_{(4)}^{(\bar{1})}h_{(5)}^{(\bar{1})} \otimes Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})h_{(5)}^{(\bar{2})}\Phi(S^2h_{(2)}Sh_{(1)}) \\ &= Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})} \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})h_{(3)}^{(\bar{2})}, \end{aligned}$$

where we used the fact that $h \triangleright m = \Phi(h_{(1)})m\Phi(Sh_{(2)})$, $\forall m \in M, h \in H$. Therefore

$$1_H \otimes \Phi(Sh) = Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})} \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})h_{(3)}^{(\bar{2})}. \quad (15)$$

Similarly, for any $h \in H$

$$\epsilon(h)1_H \otimes 1_M = h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)}). \quad (16)$$

Now we can start proving the proposition. First assume that ω_U is a strong, left-invariant connection. Recall from [8] that the connection $\Pi : \Omega^1 P \rightarrow \Omega^1 P$ in $P(M, H)$ is said to be strong if $(\text{id} - \Pi)(d_U P) \subset (\Omega^1 M)P$. In the case of a trivial bundle $P(M, H, \Phi)$ this is equivalent to the existence of a map $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 M$, given by $\beta_U(h) = \Phi(h_{(1)})\omega_U(\pi_\epsilon(h_{(2)}))\Phi^{-1}(h_{(3)}) + \Phi(h_{(1)})d_U\Phi^{-1}(h_{(2)})$. In our case Φ is an algebra map, therefore $\Phi^{-1} = \Phi \circ S$. Since ω_U is assumed to be left-invariant we find, for any $h \in \ker \epsilon_H$,

$$\begin{aligned}
& \Delta_L(\beta_U(\pi_\epsilon(h_{(1)}))h_{(2)}^{(\bar{2})}) \otimes h_{(2)}^{(\bar{1})}) \\
&= \Phi(h_{(1)})_{(1)}\Phi(Sh_{(3)})_{(1)}h_{(4)}^{(\bar{2})}_{(1)} \otimes \Phi(h_{(1)})_{(2)}\omega_U(\pi_\epsilon(h_{(2)}))\Phi(Sh_{(3)})_{(2)}h_{(4)}^{(\bar{2})}_{(2)} \otimes h_{(4)}^{(\bar{1})} \\
&\quad + \Phi(h_{(1)})_{(1)}\Phi(Sh_{(2)})_{(1)}h_{(3)}^{(\bar{2})}_{(1)} \otimes \Phi(h_{(1)})_{(2)}(d_U\Phi(Sh_{(2)})_{(2)})h_{(3)}^{(\bar{2})}_{(2)} \otimes h_{(3)}^{(\bar{1})} \\
&= \Phi(h_{(1)})^{(\bar{1})}Sh_{(5)}^{(\bar{1})})h_{(6)}^{(\bar{2})}_{(1)} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))Sh_{(5)}^{(\bar{2})}\Phi(Sh_{(4)})h_{(6)}^{(\bar{2})}_{(2)} \otimes h_{(6)}^{(\bar{1})} \\
&\quad + \Phi(h_{(1)})^{(\bar{1})}Sh_{(4)}^{(\bar{1})})h_{(5)}^{(\bar{2})}_{(1)} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U(Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)}))h_{(5)}^{(\bar{2})}_{(2)} \otimes h_{(5)}^{(\bar{1})}.
\end{aligned}$$

On the other hand since $\beta_U(h) \in \Omega^1 M$, $\Delta_L(\beta_U(h)) \in M \otimes \Omega^1 M$, i.e. Δ_L is the coaction of M on $\Omega^1 M$. Therefore the outcome of the above calculation must be in $M \otimes \Omega^1 M \otimes H$. Applying $1_H \epsilon_M \otimes \text{id}_{\Omega^1 M} \otimes \text{id}_H$ and noting that $(\epsilon_M \otimes \text{id})(1 \otimes h)(m \otimes 1) = \epsilon(m)h$, for any $h \in H$ and $m \in M$ we find

$$\begin{aligned}
& 1_H \otimes \beta_U(\pi_\epsilon(h_{(1)}))h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} \\
&= h_{(1)}^{(\bar{1})}Sh_{(5)}^{(\bar{1})} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))Sh_{(5)}^{(\bar{2})}\Phi(Sh_{(4)})h_{(6)}^{(\bar{2})} \otimes h_{(6)}^{(\bar{1})} \\
&\quad + h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U(Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})h_{(5)}^{(\bar{2})}) \otimes h_{(5)}^{(\bar{1})} \\
&\quad - h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})} \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})d_Uh_{(5)}^{(\bar{2})} \otimes h_{(5)}^{(\bar{1})}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \beta_U(\pi_\epsilon(h_{(1)}))h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} \\
&= h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))Sh_{(5)}^{(\bar{2})}\Phi(Sh_{(4)})h_{(6)}^{(\bar{2})} \otimes h_{(1)}^{(\bar{1})}Sh_{(5)}^{(\bar{1})}h_{(6)}^{(\bar{1})} \\
&\quad + h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U(Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})h_{(5)}^{(\bar{2})}) \otimes h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})}h_{(5)}^{(\bar{1})} \\
&\quad - h_{(1)}^{(\bar{2})}\Phi(h_{(2)})Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})d_Uh_{(5)}^{(\bar{2})} \otimes h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})}h_{(5)}^{(\bar{1})} \\
&= h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))\Phi(Sh_{(4)}) \otimes h_{(1)}^{(\bar{1})} + h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U\Phi(Sh_{(3)}) \otimes h_{(1)}^{(\bar{1})}
\end{aligned}$$

$$\begin{aligned}
& +h_{(1)}^{(\bar{2})}\epsilon(h_{(2)}) \otimes 1 \otimes h_{(1)}^{(\bar{1})} - h_{(1)}^{(\bar{2})}\Phi(h_{(2)})Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)}) \otimes h_{(5)}^{(\bar{2})} \otimes h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})}h_{(5)}^{(\bar{1})} \\
& = h_{(1)}^{(\bar{2})}\beta_U(\pi_\epsilon(h_{(2)})) \otimes h_{(1)}^{(\bar{1})} + h^{(\bar{2})} \otimes 1 \otimes h^{(\bar{1})} - \epsilon(h_{(1)}) \otimes h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} \\
& = h_{(1)}^{(\bar{2})}\beta_U(\pi_\epsilon(h_{(2)})) \otimes h_{(1)}^{(\bar{1})} - d_U h^{(\bar{2})} \otimes h^{(\bar{1})},
\end{aligned}$$

where we used property (15) and definition of the universal differential to derive the second equality and (16) to derive the third one.

Furthermore, we find

$$\begin{aligned}
\Delta_L(\beta_U(h)) &= \Phi(h_{(1)})_{(1)}\Phi(Sh_{(3)})_{(1)} \otimes \Phi(h_{(1)})_{(2)}\omega_U(\pi_\epsilon(h_{(2)}))\Phi(Sh_{(3)})_{(2)} \\
&\quad + \Phi(h_{(1)})_{(1)}\Phi(Sh_{(2)})_{(1)} \otimes \Phi(h_{(1)})_{(2)}d_U\Phi(Sh_{(2)})_{(2)} \\
&= \Phi(h_{(1)}^{(\bar{1})}Sh_{(5)}^{(\bar{1})}) \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))Sh_{(5)}^{(\bar{2})}\Phi(Sh_{(4)}) \\
&\quad + \Phi(h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})}) \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U(Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)}))
\end{aligned}$$

Using the fact that $\Delta_L(\beta_U(h)) \in M \otimes \Omega^1 M$ and that M is invariant under Δ_R we can apply Δ_R to first factor in $\Delta_L(\beta_U(h))$ then Φ^{-1} to second factor in the resulting tensor product and multiply first two factors to obtain back $\Delta_L(\beta_U(h))$. Applying the same procedure to the right hand side of the above equality, using the fact that Φ is an intertwiner for the right coaction of H on $M \blacktriangleright \blacktriangleleft H$ as well as the properties of a counit in $M \blacktriangleright \blacktriangleleft H$ we thus find

$$\begin{aligned}
\Delta_L(\beta_U(h)) &= \epsilon(h_{(1)}^{(\bar{1})}Sh_{(5)}^{(\bar{1})})1 \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))Sh_{(5)}^{(\bar{2})}\Phi(Sh_{(4)}) \\
&\quad + \epsilon(h_{(1)}^{(\bar{1})}Sh_{(4)}^{(\bar{1})})1 \otimes h_{(1)}^{(\bar{2})}\Phi(h_{(2)})d_U(Sh_{(4)}^{(\bar{2})}\Phi(Sh_{(3)})) \\
&= 1 \otimes (\Phi(h_{(1)})\omega_U(\pi_\epsilon(h_{(2)}))\Phi(Sh_{(3)}) + \Phi(h_{(1)})d_U\Phi(Sh_{(2)})) = 1 \otimes \beta_U(h).
\end{aligned}$$

Therefore β_U is left-invariant as stated.

Conversely, let $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 M$ be a left-invariant linear map satisfying the condition in the proposition. Define $\omega_U : \ker \epsilon_H \rightarrow \Omega^1 P$ by $\omega_U(h) = \Phi(Sh_{(1)})\beta_U(\pi_\epsilon(h_{(2)}))\Phi(h_{(3)}) + \Phi(Sh_{(1)})d_U\Phi(h_{(2)})$. The map ω_U is a strong connection 1-form. We need to verify whether it is left-invariant. For any $h \in \ker \epsilon_H$ we use the left-invariance of β_U and compute

$$\begin{aligned}
\Delta_L\omega(h) &= \Phi(Sh_{(2)}^{(\bar{1})}h_{(4)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})\beta_U(\pi_\epsilon(h_{(3)}))h_{(4)}^{(\bar{2})}\Phi(h_{(5)}) \\
&\quad + \Phi(Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})d_U(h_{(3)}^{(\bar{2})}\Phi(h_{(4)}))
\end{aligned}$$

$$\begin{aligned}
&= \Phi(Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})h_{(3)}^{(\bar{2})}\beta_U(\pi_\epsilon(h_{(4)}))\Phi(h_{(5)}) \\
&\quad - \Phi(Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})(d_U h_{(3)}^{(\bar{2})})\Phi(h_{(4)}) \\
&\quad + \Phi(Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})(d_U h_{(3)}^{(\bar{2})})\Phi(h_{(4)}) \\
&\quad + \Phi(Sh_{(2)}^{(\bar{1})}h_{(3)}^{(\bar{1})}) \otimes Sh_{(2)}^{(\bar{2})}\Phi(Sh_{(1)})h_{(3)}^{(\bar{2})}d_U\Phi(h_{(4)}) \\
&= 1 \otimes (\Phi(Sh_{(1)})\beta_U(\pi_\epsilon(h_{(2)}))\Phi(h_{(3)}) + \Phi(Sh_{(1)})d_U\Phi(Sh_{(2)})),
\end{aligned}$$

where the assumption about β_U and the Leibniz rule were used in the derivation of the second equality and the property (15) in derivation of the last one. Therefore ω_U is a left-invariant connection as required.

Since $\beta_U(h)$ is a left-invariant form on M for any $h \in \ker \epsilon_H$ then the similar argument as in the proof of Proposition 3.4 yields that $\beta_U(h) = S\gamma(h)_{(1)}d_U\gamma(h)_{(2)}$ with $\gamma = (\epsilon_M \otimes \text{id}) \circ \beta_U$, a map $\ker \epsilon_H \rightarrow \ker \epsilon_M$, which is extended uniquely to H by setting $\gamma(1) = 1$. In other words $\gamma(h) = (\epsilon_M \otimes \text{id}) \circ \beta_U(\pi_\epsilon(h)) + \epsilon(h)1_M$, for any $h \in H$. Notice that $\epsilon_M(\gamma(h)) = \epsilon_H(h)$. Assuming that β_U satisfies the condition specified in the proposition and applying $\epsilon_M \otimes \text{id}$ one finds

$$(\gamma(\pi_\epsilon(h_{(1)})) + \epsilon(h_{(1)}))h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} = (\gamma(\pi_\epsilon(h_{(2)})) + \epsilon(h_{(2)})) \otimes h_{(1)},$$

i.e.

$$\gamma(h_{(1)})h_{(2)}^{(\bar{2})} \otimes h_{(2)}^{(\bar{1})} = \gamma(h_{(2)}) \otimes h_{(1)}, \quad \forall h \in H,$$

as required. Now take any map $\gamma : H \rightarrow M$, $\gamma(1) = 1$, $\epsilon_M \circ \gamma = \epsilon_H$, and such that the above condition is satisfied. Applying $(S \otimes \text{id}) \circ \Delta$ to the first factor in this equality and using definition of the universal differential one finds

$$\begin{aligned}
&Sh_{(2)}^{(\bar{2})}_{(1)}S\gamma(h_{(1)})_{(1)}(d_U\gamma(h_{(1)})_{(2)})h_{(2)}^{(\bar{2})}_{(2)} \otimes h_{(2)}^{(\bar{1})} - S\gamma(h_{(2)})_{(1)}d_U\gamma(h_{(2)})_{(2)} \otimes h_{(1)} \\
&= Sh_{(2)}^{(\bar{2})}_{(1)}d_Uh_{(2)}^{(\bar{2})}_{(2)} \otimes h_{(1)}^{(\bar{1})},
\end{aligned}$$

or, by using the form of β_U , i.e. $\beta_U(h) = S\gamma(h)_{(1)}d_U\gamma(h)_{(2)}$

$$Sh_{(2)}^{(\bar{2})}_{(1)}\beta_U(\pi_\epsilon(h_{(1)}))h_{(2)}^{(\bar{2})}_{(2)} \otimes h_{(2)}^{(\bar{1})} - \beta_U(\pi_\epsilon(h_{(2)})) \otimes h_{(1)} = Sh_{(2)}^{(\bar{2})}_{(1)}d_Uh_{(2)}^{(\bar{2})}_{(2)} \otimes h_{(1)}^{(\bar{1})}.$$

By applying the coaction α to the second factor in the above equality, interchanging third factor with the second and the first ones and then multiplying first two factors one obtains the required property of β_U . Hence the bijective correspondence between β_U and γ is established.

Finally, from Proposition 3.4, left-invariant ω_U is of the canonical form with $i = (\epsilon \otimes \text{id}) \circ \omega_U$. Since $\omega_U(h) = \Phi(Sh_{(1)})\beta_U(\pi_\epsilon(h_{(2)}))\Phi(h_{(3)}) + \Phi(Sh_{(1)})d_U\Phi(h_{(2)})$ one easily finds that $i(h) = \gamma(h_{(1)})\Phi(h_{(2)})$, i.e. $i(h) = \gamma(h_{(1)}) \otimes h_{(2)}$, where $\gamma : H \rightarrow M$, $\gamma(h) = (\epsilon \otimes \text{id}) \circ \beta(\pi_\epsilon(h)) + \epsilon(h)1_M$, for any $h \in H$. \square

Therefore, for these β_U we are in the setting of Proposition 3.4 or Example 3.6 for the map i constructed above. The smallest horizontal right ideal in this case is

$$\mathcal{Q}_0 = \text{span}\{\gamma(q_{(1)})q_{(2)}\triangleright m \otimes q_{(3)}h - \epsilon(m)\gamma(q_{(1)}h_{(1)}) \otimes q_{(2)}h_{(2)} \mid q \in \mathcal{Q}, m \in M, h \in H\}. \quad (17)$$

We see that a choice of left-invariant ω_U , $\mathcal{Q}_{\text{hor}} \supseteq \mathcal{Q}_0$ and left-covariant $\Omega^1(M)$ defines a left-covariant $\Omega^1(P)$. The corresponding ideal is $\mathcal{Q}_P = \langle i(\mathcal{Q})P, \mathcal{Q}_M P \rangle$ where

$$i(\mathcal{Q})P = \text{span}\{\gamma(q_{(1)})q_{(2)}\triangleright m \otimes q_{(3)}h \mid q \in \mathcal{Q}, m \in M, h \in H\} \supseteq \mathcal{Q}_0.$$

Example 6.2 Let $P = M \blacktriangleright \triangleleft H$ be viewed as a quantum principal bundle. Let γ obey the condition in Proposition 6.1 and let $\Omega^1(M)$ be left M -covariant. Then P has a natural left-covariant calculus $\Omega^1(P)$ such that $\Omega_{\text{hor}}^1 = P(dM)P$ and

$$\omega(h) = \Phi^{-1}(h_{(1)})\beta(\pi_\epsilon(h_{(2)}))\Phi(h_{(3)}) + \Phi^{-1}(h_{(1)})d\Phi(h_{(2)})$$

where $\beta : \ker \epsilon \rightarrow \Omega_P^1(M)$ is defined by $\beta(h) = (S\gamma(h)_{(1)})d\gamma(h)_{(2)}$.

Proof Since $\Omega^1(M)$ is assumed to be left-covariant, the subbimodule \mathcal{N}_M generating $\Omega^1(M)$ is obtained from a right ideal $\mathcal{Q}_M \subset \ker \epsilon_M$. Since M is a Hopf subalgebra of $M \blacktriangleright \triangleleft H$, the left M -invariance of \mathcal{N}_M implies left P -invariance of $P\mathcal{N}_M P$. The corresponding right ideal in $\ker \epsilon_P$ is $\mathcal{Q}_M P$. Therefore we take $\mathcal{Q}_{\text{hor}} = \langle \mathcal{Q}_0, \mathcal{Q}_M P \rangle$ corresponding to $\mathcal{N}_{\text{hor}} = \langle \mathcal{N}_0, P\mathcal{N}_M P \rangle$ as in Example 3.7. On the other hand, we are also in the setting of Proposition 3.3 and take ω_U in that strong form, as in Proposition 6.1. Note as in Proposition 3.3. that the inherited differential structure $\Omega_P^1(M)$ is not different from $\Omega^1(M)$ if $\mathcal{N}_0 \cap \Omega^1 M \subseteq \mathcal{N}_M$. \square

We now consider the simplest concrete setting of bicrossproducts, where $M = \mathbb{C}(\Sigma)$, Σ a finite set (as in Section 5) and $H = \mathbb{C}G$, G a finite group. Here P is a bicrossproduct of the

form $\mathbb{C}(\Sigma) \blacktriangleright \mathbb{C}G$, regarded as a bundle. This is necessarily of the form associated to a group factorisation $X = G\Sigma$. Then Σ acts on G and G acts on Σ , by $\triangleright, \triangleleft$ respectively, as defined by $sg = (s \triangleright g)(s \triangleleft g)$ in X . The bicrossproduct $\mathbb{C}(\Sigma) \blacktriangleright \mathbb{C}G$ has the explicit form

$$\begin{aligned} (\delta_s \otimes g)(\delta_t \otimes h) &= \delta_{s \triangleleft g, t}(\delta_s \otimes gh) \\ \Delta(\delta_s \otimes g) &= \sum_{ab=s} \delta_a \otimes b \triangleright g \otimes \delta_b \otimes g, \quad S(\delta_s \otimes g) = \delta_{(s \triangleleft g)^{-1}} \otimes (s \triangleright g)^{-1} \end{aligned}$$

for all $g, h \in G$ and $s, t \in \Sigma$. Note that the actions $\triangleright, \triangleleft$ are typically not effective. We define the subset

$$Y = \{(g, s) \mid s \triangleright g = g\} = \prod_{g \in G} I(g) \subseteq G \times \Sigma.$$

where $I(g)$ is the isotropy group of g . Here Y necessarily contains $\Sigma = I(e)$ as (e, Σ) where $e \in G$ is the group identity. From Proposition 5.1 we know that $\Omega^1(\mathbb{C}(\Sigma))$ correspond to $\Gamma \subset \Sigma \times \Sigma$ -diag. We require this to be Σ -invariant. Finally, we know from [13] that coirreducible bicovariant $\Omega^1(\mathbb{C}G)$ correspond to (V, λ) where V is an irreducible left G -module and $\lambda \in P(V^*)$. The corresponding quantum tangent space in [13] is spanned by $x_v = \lambda((\) \triangleright v) - \lambda(v)1 \in \mathbb{C}(G)$ with corresponding derivation $\partial_{x_v} g = x_v(g)g$ on group-like elements $g \in \mathbb{C}G$. Hence

$$\mathcal{Q} = \text{span}\{q \in \ker \epsilon \mid \epsilon \partial_{x_v} q = 0\} = \{q \in \ker \epsilon \mid \lambda(q \triangleright v) = 0 \quad \forall v \in V\}$$

i.e. the kernel of the map $\ker \epsilon \rightarrow V^*$ provided by the action $\triangleright : \mathbb{C}G \otimes V \rightarrow V$ composed with λ .

Proposition 6.3 *Left-invariant $\Omega^1(\mathbb{C}(\Sigma) \blacktriangleright \mathbb{C}G)$ are provided by pairs γ, S , where $\gamma \in \mathbb{C}(Y)$ is a function such that $\gamma(e, s) = 1 = \gamma(g, e)$ for all $s \in \Sigma, g \in G$, and $S \subset \Sigma$, $e \notin S$ is a subset. The associated invariant connection is defined by*

$$\beta_U(g)_{s,t} = \gamma(g, s^{-1}t) - 1$$

where γ is extended by zero to X . The minimal horizontal right ideal is

$$\mathcal{Q}_0 = \text{span}\left\{\sum_{q \in G} q_g \gamma(g) \delta_{s \triangleleft g^{-1}} \otimes gh \mid q \in \mathcal{Q}, h \in G, e \neq s \in \Sigma\right\} + \text{span}\left\{\sum_{g \in G} q_g (\delta_e - \gamma(g)) \otimes g \mid q \in \mathcal{Q}\right\}.$$

If we take $\mathcal{Q}_{\text{hor}} = \langle \mathcal{Q}_0, \mathbb{C}(S) \otimes \mathbb{C}G \rangle$ then the resulting calculus has

$$\mathcal{Q}_P = \text{span}\left\{\sum_{q \in G} q_g \gamma(g) \delta_{s \triangleleft g^{-1}} \otimes gh \mid q \in \mathcal{Q}, h \in G, e \neq s \in \Sigma\right\} + \delta_e \otimes \mathcal{Q} + \mathbb{C}(S) \otimes \mathbb{C}G.$$

Proof The coaction of $\mathbb{C}(\Sigma)$ on \mathcal{G} in the bicrossproduct is $g \mapsto \sum_s s \triangleright g \otimes \delta_s$ (see [10]). We therefore require $\gamma : \mathbb{C}G \rightarrow \mathbb{C}(\Sigma)$ i.e. $\gamma \in \mathbb{C}(G \times \Sigma)$ such that

$$\sum_s \gamma(g) \delta_s \otimes s \triangleright g = \gamma(g) \otimes g$$

for all g . Evaluating at a fixed $s \in \Sigma$, this is $\gamma(g, s)(s \triangleright g - g) = 0$ for all $s \in \Sigma$ and $g \in G$. This gives the stated form of γ . Then

$$\begin{aligned} \beta_U(g) &= \sum_{s \in I(g)} \gamma(g, s) \sum_{ab=s} \delta_{a^{-1}} d_U \delta_b = \sum_{s \in I(g)} \gamma(g, s) \sum_{ab=s} \delta_{a^{-1}} \otimes \delta_b - \delta_{a^{-1}} \delta_b \otimes 1 \\ &= \sum_{s \in I(g)} \gamma(g, s) \sum_{ab=s} \delta_{a^{-1}} \otimes \delta_b - \gamma(g, e) 1 \otimes 1 \end{aligned}$$

which gives the formula for components of β_U as stated.

Next, we require $\Omega^1(\mathbb{C}(\Sigma))$ to be left $\mathbb{C}(\Sigma)$ -invariant, i.e. that $\Delta_L \mathcal{N}_M \subset M \otimes \mathcal{N}_M$ where $M = \mathbb{C}(\Sigma)$ has coproduct $\Delta \delta_s = \sum_{ab=s} \delta_a \otimes \delta_b$. As in Section 5 we take $\mathcal{N}_M = \text{span}\{\delta_s \otimes \delta_t \mid (s, t) \in \Gamma\}$. The invariance is then equivalent to Γ stable under the diagonal action of the group Σ . Such Γ are of the form $\Gamma = \{(s, t) \mid s^{-1}t \in S\}$ for some subset S not containing the group identity e . The right ideal \mathcal{Q}_M in this case is

$$\mathcal{Q}_M = \text{span}\{\delta_s \mid s \in S\} = \mathbb{C}(S).$$

From the form of the algebra structure of the bicrossproduct, it is clear that $\mathcal{Q}_M P = \mathbb{C}(S) \otimes \mathbb{C}G$.

To compute \mathcal{Q}_0 we consider elements in \mathcal{Q} of the form $q = \sum_{g \in G} q_g g$ and the delta-function basis for $\mathbb{C}(\Sigma)$ in the formula (17). Then

$$\begin{aligned} \mathcal{Q}_0 &= \text{span}\left\{ \sum_{g \in G} q_g (\gamma(g) g \triangleright \delta_s - \delta_{s,e} \gamma(gh)) \otimes gh \mid q \in \mathcal{Q}, h \in G, s \in \Sigma \right\} \\ &= \text{span}\left\{ \sum_{g \in G} q_g \gamma(g) \delta_{s \triangleleft g^{-1}} \otimes gh \mid q \in \mathcal{Q}, h \in G, e \neq s \in \Sigma \right\} \\ &\quad + \text{span}\left\{ \sum_{g \in G} q_g (\delta_e - \gamma(g)) \otimes g \mid q \in \mathcal{Q} \right\} \end{aligned}$$

where we consider the cases where $s = e$ and $s \neq e$ separately. In the former part we wrote $g \triangleright \delta_s = \delta_{s \triangleleft g^{-1}}$ while in the case $s = e$ we change variables from $qh = \sum_g q_g gh$ to q since $qh \in \mathcal{Q}$ for all h . We span over $q \in \mathcal{Q}$ after fixing $s \in M$ and $h \in G$. The computation of \mathcal{Q}_P is similar.

□

We demonstrate this construction now in some examples based on finite cyclic groups. When $G = \mathbb{Z}_n = \langle g \rangle$, for the representation V defining a calculus on $\mathbb{C}G$ we take the 1-dimensional representation where the generator g acts as $e^{\frac{2\pi i}{n}}$. Its character χ corresponds to a conjugacy class in $\hat{\mathbb{Z}}_n$ if we take the view $\mathbb{C}\mathbb{Z}_n \cong \mathbb{C}(\hat{\mathbb{Z}}_n)$. The corresponding quantum tangent space is spanned by $x = \chi - 1 \in \mathbb{C}(\mathbb{Z}_n)$ with corresponding derivation $\partial_x g^a = x(g^a)g^a$ for $a \in \{0, \dots, n-1\}$. Hence

$$\mathcal{Q} = \{q \in \mathbb{C}G \mid \epsilon(q) = 0, \chi(q) = 0\} = \{q_a = \sum_{b=0}^{n-1} e^{\frac{2\pi i ab}{n}} g^b \mid a = 1, 2, 3, n-2\}$$

The remaining basis element $n^{-1}q_{n-1}$ of $\ker \epsilon$ is dual to x and can be identified with the unique normalised left-invariant 1-form in the calculus.

Likewise, for a calculus on $\mathbb{C}(\Sigma)$ where $\Sigma = \mathbb{Z}_m = \langle s \rangle$ we take for left-invariant calculus the one defined by $S = \{s^2, s^3, \dots, s^{m-1}\}$. Since Σ is Abelian, left-invariant calculi are automatically bicovariant, and this is the natural 1-dimensional bicovariant calculus $\Omega^1(\mathbb{C}(\Sigma))$ associated to the generator $s \in \Sigma$. The ideal \mathcal{Q}_M consists of all functions vanishing at e, s . The element δ_s is dual to the quantum tangent space basis element $s - e$ and can be identified with the unique normalised left-invariant 1-form.

There are many factorisations of the form $\mathbb{Z}_n \mathbb{Z}_m$. We consider one of the simplest, namely $S_3 = \mathbb{Z}_2 \mathbb{Z}_3$ (actually a semidirect product) where $G = \mathbb{Z}_2 = \langle g \rangle$ and $\Sigma = \mathbb{Z}_3 = \langle s \rangle$. In terms of permutations α, β obeying $\alpha^2 = \beta^2 = e$ and $\alpha\beta\alpha = \beta\alpha\beta$, we write $g = \alpha$ and $s = \alpha\beta$. The action \triangleright is trivial while $s \triangleleft g = s^2$ and $s^2 \triangleleft g = s$. The Hopf algebra $\mathbb{C}(Z_3) \rtimes \mathbb{C}\mathbb{Z}_2$ is 6-dimensional with cross relations

$$g\delta_e = \delta_e g, \quad g\delta_s = \delta_{s^2} g, \quad g\delta_{s^2} = \delta_s g$$

and the tensor product coalgebra structure. The subset Y is all of $G \times \Sigma$ and hence

$$\gamma(e) = 1, \quad \gamma(g) = \delta_e + \gamma_1 \delta_s + \gamma_2 \delta_{s^2}$$

for two parameters $\gamma_1, \gamma_2 \in \mathbb{C}$.

Example 6.4 For the cross product $P = \mathbb{C}(Z_3) \rtimes \mathbb{C}\mathbb{Z}_2$ as above and the choice of the 1-dimensional $\Omega^1(\mathbb{C}\mathbb{Z}_2)$ and $\Omega^1(\mathbb{C}(Z_3))$ as above, we find $\Omega^1(P)$ is 3-dimensional corresponding to $\mathcal{Q}_P =$

$\text{span}\{\delta_{s^2}\} \otimes \mathbb{C}\mathbb{Z}_2$. It has basis of invariant forms $\{\omega_0, \omega_1, \omega_2\}$ say and

$$d\delta_e = (\delta_{s^2} - \delta_e)\omega_1, \quad d\delta_s = (\delta_e - \delta_s)\omega_1, \quad dg = g(\omega_0 - \omega_1 + \omega_2)$$

and module structure

$$\omega_0 g = -g\omega_0, \quad \omega_1 g = g\omega_2, \quad \omega_2 g = g\omega_1$$

$$\omega_0 \delta_{s^i} = \delta_{s^i} \omega_0, \quad \omega_1 \delta_{s^i} = \delta_{s^{i-1}} \omega_1, \quad \omega_2 \delta_{s^i} = \delta_{s^{i-1}} \omega_2.$$

The gauge field corresponding to γ is

$$\beta_U(g) = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 \\ \gamma_2 & 0 & \gamma_1 \\ \gamma_1 & \gamma_2 & 0 \end{pmatrix}$$

but the entries γ_i do not affect the resulting calculus.

Proof The ideal $\mathcal{Q} = 0$ in this case, i.e. $\Omega^1(\mathbb{C}\mathbb{Z}_2)$ is being taken here with the universal differential calculus, which is 1-dimensional in the case of $\mathbb{C}\mathbb{Z}_2$. This is clear from the point of view of a bicovariant calculus on $\mathbb{C}(\hat{\mathbb{Z}}_2)$. Hence $\mathcal{Q}_0 = 0$ as well, and we take $\mathcal{Q}_{\text{hor}} = \mathcal{Q}_M P = \text{span}\{\delta_{s^2}\} \otimes \mathbb{C}\mathbb{Z}_2$ for all γ . According to Proposition 3.5, $\mathcal{Q}_P = \langle \mathcal{Q}_{\text{hor}}, i(\mathcal{Q})P \rangle = \mathcal{Q}_M P$ as well since $\mathcal{Q} = 0$. This gives the calculus $\Omega^1(P)$. It projects to the universal one in the fibre direction and restricts to the initial calculus on the base.

We now compute this 3-dimensional calculus explicitly. We recall that $\Omega^1(P) = P \otimes \ker \epsilon / \mathcal{Q}_P$ as a left P -module by multiplication by P , as a right P module by $[h]u = u_{(1)} \otimes [hu_{(2)}]$ for $[h] \in \ker \epsilon / \mathcal{Q}_P$ and $u \in P$. Here $[\]$ denotes the canonical projection from $\ker \epsilon$. The exterior derivative is $du = u_{(1)} \otimes u_{(2)} - u \otimes 1$ projected to $\ker \epsilon / \mathcal{Q}_P$. In our case, a basis for the latter is

$$\omega_0 = [\delta_e \otimes (g - e)], \quad \omega_1 = [\delta_s \otimes e], \quad \omega_2 = [\delta_s \otimes g].$$

Then $d(1 \otimes g) = 1 \otimes g \otimes [1 \otimes (g - e)]$ giving the result as stated on identifying $g \equiv 1 \otimes g$ in P . Moreover, $d(\delta_e \otimes e) = \delta_e \otimes e \otimes \delta_e \otimes e + \delta_{s^2} \otimes e \otimes \delta_s \otimes e - \delta_e \otimes e \otimes 1 \otimes e = (\delta_{s^2} \otimes e - \delta_e \otimes e) \otimes [\delta_s \otimes e]$ as stated, on identifying $\delta_s \otimes e \equiv \delta_s$ etc. Likewise, $d(\delta_s \otimes e) = \delta_s \otimes e \otimes \delta_e \otimes e + \delta_e \otimes e \otimes \delta_s \otimes e - \delta_s \otimes e \otimes 1 \otimes e = (\delta_e \otimes e - \delta_s \otimes e) [\delta_s \otimes e]$ as stated.

Finally, we compute the right module structure as follows. For the action on ω_0 we have $[\delta_e \otimes (g - e)](1 \otimes g) = 1 \otimes g \otimes [(\delta_e \otimes (g - e))(1 \otimes g)] = -1 \otimes g \otimes [\delta_e \otimes (g - e)]$ as stated. And

$[\delta_e \otimes (g - e)](\delta_{s^i} \otimes e) = \sum_{a+b=i} \delta_{s^a} \otimes e \otimes [(\delta_e \otimes (g - e))(\delta_{s^b} \otimes e)] = \delta_{s^i} \otimes e \otimes [\delta_e \otimes (g - e)]$ as stated.

Only the $b = 0$ term in the sum contributes. For the action on ω_1 we have $[\delta_s \otimes e](1 \otimes g) = 1 \otimes g \otimes [(\delta_s \otimes g)(1 \otimes g)] = [\delta_s \otimes e]$. And $[\delta_s \otimes e](\delta_{s^i} \otimes e) = \sum_{a+b=i} \delta_{s^a} \otimes e \otimes [(\delta_s \otimes e)(\delta_{s^b} \otimes e)] = \delta_{s^{i-1}} \otimes e \otimes [\delta_s \otimes e]$ as only the $b = 1$ term in the sum contributes. Similarly for the action on ω_2 .

As a left module the action is free, i.e. we identify $(1 \otimes g) \otimes \omega_0 = g\omega_0$ etc. \square

Example 6.5 For the cross product $P = \mathbb{C}(\mathbb{Z}_3) \rtimes \mathbb{C}\mathbb{Z}_2$ as above but the choice of zero differential calculus $\Omega^1(\mathbb{C}\mathbb{Z}_2)$ and universal calculus $\Omega^1\mathbb{C}(\mathbb{Z}_3)$, we find $\Omega^1(P)$ is the zero calculus unless $\gamma_1\gamma_2 = 1$, when it is 2-dimensional. In the latter case, with basis of invariant forms $\{\omega_1, \omega_2\}$ we have

$$d\delta_e = (\delta_{s^2} - \delta_e)\omega_1 + \gamma_1(\delta_s - \delta_e)\omega_2, \quad d\delta_s = (\delta_e - \delta_s)\omega_1 + \gamma_1(\delta_{s^2} - \delta_s)\omega_2, \quad dg = (1 - \gamma_1)g(\omega_2 - \omega_1)$$

and right module structure

$$\omega_1 g = g\omega_2, \quad \omega_2 g = g\omega_1, \quad \omega_1 \delta_{s^i} = \delta_{s^{i-1}}\omega_1, \quad \omega_2 \delta_{s^i} = \delta_{s^{i-1}}\omega_2.$$

The restriction to $\Omega_P^1(\mathbb{C}(\mathbb{Z}_3))$ is a direct sum of the 1-dimensional calculus associated to s and the 1-dimensional calculus associated to s^2 .

Proof If we take the zero differential calculus on $\Omega^1(\mathbb{C}\mathbb{Z}_2)$, so $\mathcal{Q} = \mathbb{C}(g - e)$, then

$$\mathcal{Q}_0 = \text{span}\{\gamma_1\delta_s \otimes gh - \delta_{s^2} \otimes h, \gamma_2\delta_{s^2} \otimes gh - \delta_s \otimes h \mid h \in \mathbb{Z}_2\}.$$

Here the s contribution to \mathcal{Q}_0 is $\gamma(g)\delta_{s^2} \otimes gh - \gamma(e)\delta_s \otimes h = \gamma_2\delta_{s^2} \otimes gh - \delta_s \otimes h$. Similarly, the s^2 contribution is $\gamma_1\delta_s \otimes gh - \delta_{s^2} \otimes h$. Finally, the s^0 contribution is $(\delta_e - \gamma(g)) \otimes g - (\delta_e - 1) \otimes e = -\gamma_1\delta_s \otimes g + \delta_{s^2} \otimes e - \gamma_2\delta_{s^2} \otimes g + \delta_s \otimes e$ is already contained. This is 4-dimensional for generic γ (in this case $\mathcal{Q}_0 = \ker \epsilon \otimes \mathbb{C}\mathbb{Z}_2$) but collapses to a 2-dimensional ideal when $\gamma_1\gamma_2 = 1$.

If we take the universal calculus on $\mathbb{C}(\mathbb{Z}_3)$ so $\mathcal{Q}_M = 0$, we have $\mathcal{Q}_{\text{hor}} = \mathcal{Q}_0$ is 4-dimensional in the generic case or 2-dimensional in the degenerate case (note that if we took the 1-dimensional calculus on $\mathbb{C}(\mathbb{Z}_3)$ as before then $\mathcal{Q}_{\text{hor}} = \ker \epsilon \otimes \mathbb{C}\mathbb{Z}_2$ is 4-dimensional in either case). Finally, $i(\mathcal{Q}) = \gamma(g) \otimes g - 1 \otimes e = \delta_e \otimes (g - e) + \gamma_1\delta_s \otimes g - \delta_{s^2} \otimes e + \gamma_2\delta_{s^2} \otimes g - \delta_s \otimes e$ so $\mathcal{Q}_P = \langle \mathcal{Q}_{\text{hor}}, i(\mathcal{Q})P \rangle = \ker \epsilon \otimes \mathbb{C}\mathbb{Z}_2 + \mathbb{C}\delta_e \otimes (g - e) = \ker \epsilon$ is 5-dimensional except in the degenerate

case when $\gamma_1\gamma_2 = 1$. This means that the calculus on P is the zero one except in the degenerate case. In the degenerate case, $\mathcal{Q}_P = \text{span}\{q, qg, \delta_e \otimes (g - e)\}$ is 3-dimensional, where $q = \gamma_1\delta_s \otimes g - \delta_{s^2} \otimes e$ as a shorthand.

In this degenerate case, a basis of $\ker \epsilon / \mathcal{Q}_P$ is

$$\omega_1 = [\delta_s \otimes e], \quad \omega_2 = [\delta_s \otimes g]$$

while in this quotient, $\delta_{s^2} \otimes e = \gamma_1\delta_s \otimes g$ and $\delta_{s^2} \otimes g = \gamma_1\delta_s \otimes e$ instead of zero as in the preceding example, while $\delta_e \otimes (g - e)$ is now zero in the quotient. The computations proceed as on the preceding example with these changes, resulting in some extra terms with γ_1 as stated. The restriction to $\mathbb{C}(\mathbb{Z}_3)$ has a part spanned by ω_1 which is the 1-dimensional calculus on $\mathbb{C}(\mathbb{Z}_3)$ as in the preceding example and a part spanned by ω_2 which has a similar form when computed for $d\delta_{s^2}$. \square

For a more complicated example one may take $S_3 \times S_3 = \mathbb{Z}_6 \bowtie \mathbb{Z}_6$ in [32], which is a genuine double cross product with both $\triangleright, \triangleleft$ nontrivial. Writing $G = \mathbb{Z}_6 = \langle g \rangle$ and $\Sigma = \mathbb{Z}_6 = \langle s \rangle$, say, the actions of the generators are by group inversion on the other group. Thus

$$I(e) = I(g^3) = \Sigma, \quad I(g) = I(g^2) = I(g^4) = I(g^5) = \{e, s^2, s^4\} = \mathbb{Z}_3.$$

The space of allowed γ is therefore 13-dimensional. The bicrossproduct Hopf algebra $P = \mathbb{C}(\mathbb{Z}_6) \blacktriangleright \mathbb{C}Z_6$ in this case is 36-dimensional. The results in this case are similar to the situation above: for generic parameters one obtains the zero calculus but for special values one obtains calculi on $\Omega^1(P)$ restricting to non-universal calculi on the base.

Finally, one may apply Proposition 6.2 equally well in the setting of Lie bicrossproducts. As shown in [33] one has examples $\mathbb{C}(G^{\star\text{op}}) \blacktriangleright U(g)$ for all simple Lie algebras g . Here $G^{\star\text{op}}$ is the solvable group in the Iwasawa decomposition of the complexification of the compact Lie group G with Lie algebra g . Such bicrossproduct quantum groups arise as the actual algebra of observables of quantum systems, for example the Lie bicrossproduct $\mathbb{C}(SU_2^{\star\text{op}}) \blacktriangleright U(su_2)$ is the quantum algebra of observables of a deformed top[34][10]. We consider this example briefly. We take $\mathbb{C}(SU_2^{\star\text{op}})$ as described by coordinates $\{X_i\}$ and $(X_3 + 1)^{-1}$ adjoined, and a usual basis $\{e_i\}$

of su_2 . Then the bicrossproduct is (see [10])

$$[X_i, X_j] = 0, \quad \Delta X_i = X_i \otimes 1 + (X_3 + 1) \otimes X_i, \quad \epsilon X_i = 0, \quad S X_i = -\frac{X_i}{X_3 + 1}.$$

$$\begin{aligned} [e_i, e_j] &= \epsilon_{ijk} e_k, \quad [e_i, X_j] = \epsilon_{ijk} X_k - \frac{1}{2} \epsilon_{ij3} \frac{X^2}{X_3 + 1}, \quad \epsilon e_i = 0, \\ \Delta e_i &= e_i \otimes \frac{1}{X_3 + 1} + e_3 \otimes \frac{X_i}{X_3 + 1} + 1 \otimes e_i, \quad S e_i = e_3 X_i - e_i (X_3 + 1). \end{aligned}$$

For a differential calculus $\Omega^1(\mathbb{C}(SU_2^{\star\text{op}}))$ we have a range of choices including the standard commutative one. Others are ones with quantum tangent space given by jet bundles[13]. For $\Omega^1(U(su_2))$ one may follow a similar prescription to $\Omega^1(\mathbb{C}G)$: if V is an irreducible representation and $\lambda \in P(V^*)$ then $\mathcal{Q} = \{q \in U(su_2) \mid \epsilon(q) = 0, \lambda(q \triangleright v) = 0, \forall v \in V\}$. A natural choice is V a highest weight representation and λ the conjugate to the highest weight vector. Finally, we consider the possible γ . Note first of all that in a von-Neumann algebra setting one may consider group elements $g \in SU_2$ much as in Proposition 6.3. From the explicit formulae for the action of $su_2^{\star\text{op}}$ on SU_2 in [33][10], one then sees that at least near the group identity,

$$I(g) = \{\exp t(f_3 - \text{Rot}_g(f_3)) \mid t \in \mathbb{R}\}$$

where $\{f_i\}$ are the associated basis of the Lie algebra $su_2^{\star\text{op}}$ and Rot is the action of SU_2 by rotations of \mathbb{R}^3 . Hence γ should be some form of distribution on $SU_2 \times SU_2^{\star\text{op}}$ such that $\gamma(g)$ has support in the line $I(g)$. This suggests that in our algebraic setting one should be able to construct a variety of $\gamma : U(su_2) \rightarrow \mathbb{C}(SU_2^{\star\text{op}})$ order by order in a basis of $U(su_2)$. Thus, at the lowest order the coaction of $\mathbb{C}(SU_2^{\star\text{op}})$ is[10]

$$e_i^{(\bar{1})} \otimes e_i^{(\bar{2})} = e_i \otimes (X_3 + 1)^{-1} + e_3 \otimes X_i (X_3 + 1)^{-1}$$

and the condition for γ in Proposition 6.2 becomes

$$\gamma(e_i) X_3 - \gamma(e_3) X_i = 0.$$

This has solutions of the form $\gamma(e_i) = f_i(X) X_i$ for any functions $f_i \in \mathbb{C}(SU_2^{\star\text{op}})$. After fixing γ and the base and fibre differential calculi one may obtain left-invariant differential calculus on $P = \mathbb{C}(SU_2^{\star\text{op}}) \blacktriangleleft U(su_2)$ and a connection on it as a quantum principal bundle. The detailed analysis will be considered elsewhere.

We note that as a semidirect product one could also think of this bicrossproduct as a deformation of the 3-dimensional Euclidean group of motions (one may introduce a scaling of the X_i to achieve this). In the 3+1 dimensional version of this same construction one has the κ -deformed Poincaré algebra as such a bicrossproduct[35]. Proposition 6.2 therefore provides in principle a general construction for left-invariant calculi on these as well. At the moment, only some examples are known by hand [36]. Moreover, affine quantum groups such as $U_q(\hat{su}_2)$ may be considered as cocycle bicrossproducts $\mathbb{C}[c, c^{-1}] \blacktriangleright U_q(Lsu_2)$ where $U_q(Lsu_2)$ is the level zero affine quantum group (quantum loop group) and c is the central charge generator, see [37]. The quantum Weyl groups provide still more examples of cocycle bicrossproducts[38]. All of these and their duals may be treated as (trivial) quantum principal bundles by similar methods to those above.

6.2 Biproducts, bosonisations and the quantum double

Let H be a Hopf algebra with (for convenience) bijective antipode. A braided group in the category of crossed modules means B which is an algebra, a coalgebra and a crossed H -module (i.e. a left H -module and left H -comodule in a compatible way) with all structure maps intertwining the action and coaction of H and with the coproduct $\underline{\Delta} : B \rightarrow B \underline{\otimes} B$ a homomorphism in the braided tensor product algebra structure $B \underline{\otimes} B$. This is basically the same thing as a braided group in the category of $D(H)$ -modules where $D(H)$ is Drinfeld double in the finite-dimensional case. One knows from the braided setting[5] of [31] that every such braided group has an associated Hopf algebra $B \bowtie H$ as cross product and cross coproduct. Moreover, $\pi(b \otimes h) = \underline{\epsilon}(b)h$ defines a projection $B \bowtie H \rightarrow H$ split by Hopf algebra map $j(h) = 1 \otimes h$. All split Hopf algebra projections to H are of this form.

We can clearly view such $B \bowtie H$ as principal bundles of the homogeneous type, with j defining a canonical connection[1]. The homogeneous bundle coaction is $\Delta_R(b \otimes h) = b \otimes h_{(1)} \otimes h_{(2)}$ so that $M = B$. Since j is a coalgebra map we can also take $\Phi = j$ as a trivialisation, i.e. the bundle is trivial. Both Propositions 3.3 and 3.5 apply in this case. From the former, we know that any $\beta_U : \ker \epsilon \rightarrow \Omega^1 B$ and any $\Omega^1(H)$ yields a calculus $\Omega^1(B \bowtie H)$. We now use Proposition 3.5 to study which of these are left-invariant.

Note that B as a braided group coacts on itself via the braided coproduct. This is the braided left regular coaction. This extends to $B \underline{\otimes} B$ as a braided tensor product coaction, via the braiding $\Psi(v \otimes w) = v^{(\bar{1})} \triangleright w \otimes v^{(\bar{2})}$ of the category of crossed modules. So

$$\underline{\Delta}_L(b \otimes c) = \underline{b}_{(1)} \Psi(\underline{b}_{(2)} \otimes \underline{c}_{(1)}) \otimes \underline{c}_{(2)} = \underline{b}_{(1)} (\underline{b}_{(2)}^{(\bar{1})} \triangleright \underline{c}_{(1)}) \otimes \underline{b}_{(2)}^{(\bar{2})} \otimes \underline{c}_{(2)}$$

and this restricts to a left B -coaction on $\Omega^1(B)$. The calculus $\Omega^1(B)$ is braided-left covariant if its associated ideal \mathcal{N}_B is stable under $\underline{\Delta}_L$.

Proposition 6.6 *Strong left-invariant connections ω_U on $B \rtimes H(B, H, j)$ are in 1-1 correspondence with the maps $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 B$ which are left B -invariant (under the braided coproduct) and intertwine the left H -coaction on $\Omega^1 B$ with the left-adjoint coaction of H , i.e.*

$$\Delta_H(\beta_U(h)) = h_{(1)} Sh_{(3)} \otimes \beta_U(h_{(2)}).$$

Moreover, the β_U are in 1-1 correspondence with $\gamma : \ker \epsilon_H \rightarrow \ker \underline{\epsilon}$ which are intertwiners of the left adjoint coaction of H , i.e.

$$\Delta_H(\gamma(h)) = h_{(1)} Sh_{(3)} \otimes \gamma(h_{(2)}).$$

The correspondence is via

$$\beta_U(h) = \underline{S}\gamma(h)_{(1)} \underline{d}_U \gamma(h)_{(2)}.$$

The corresponding ω_U is the canonical connection for the splitting $i(h) = \gamma(\pi_\epsilon(h_{(1)})) \otimes h_{(2)} + 1 \otimes h$

Proof $B \rtimes H(B, H, j)$ is a trivial bundle with trivialisation j . Given strong connection ω_U one associates to it the unique map $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 B$ given by $\beta_U(h) = j(h_{(1)})\omega_U(\pi_\epsilon(h_{(2)}))j(Sh_{(3)}) + j(h_{(1)})\underline{d}_U j(Sh_{(2)})$. Since j is a Hopf algebra map the left coaction Δ_L of $B \rtimes H$ on $\beta_U(h) \in \Omega^1 B \in \Omega^1 B \rtimes H$ can be easily computed using the fact the ω_U is left-invariant

$$\begin{aligned} \Delta_L(\beta_U(h)) &= j(h_{(1)})j(Sh_{(5)}) \otimes j(h_{(2)})\omega_U(\pi_\epsilon(h_{(3)}))j(Sh_{(4)}) + j(h_{(1)})j(Sh_{(4)}) \otimes j(h_{(2)})\underline{d}_U j(Sh_{(3)}) \\ &= j(h_{(1)}Sh_{(3)}) \otimes \beta_U(h_{(2)}) = 1 \otimes h_{(1)}Sh_{(3)} \otimes \beta_U(h_{(2)}), \end{aligned}$$

where we also used the fact that $\ker \epsilon_H$ is invariant under the left adjoint coaction. On the other hand the left coaction of $B \bowtie H$ on $B \otimes B \subset (B \bowtie H)^{\otimes 2}$ is

$$\begin{aligned}\Delta_L(b \otimes c) &= (\underline{b}_{(1)} \otimes \underline{b}_{(2)}^{(\bar{1})})(\underline{c}_{(1)} \otimes \underline{c}_{(2)}^{(\bar{1})}) \otimes \underline{b}_{(2)}^{(\bar{2})} \otimes \underline{c}_{(2)}^{(\bar{2})} \\ &= \underline{b}_{(1)}(\underline{b}_{(2)}^{(\bar{1})} \triangleright \underline{c}_{(1)}) \otimes \underline{b}_{(2)}^{(\bar{1})} \underline{c}_{(2)}^{(\bar{1})} \otimes \underline{b}_{(2)}^{(\bar{2})} \otimes \underline{c}_{(2)}^{(\bar{2})} \\ &= \underline{b}_{(1)}(\underline{b}_{(2)}^{(\bar{1})} \triangleright \underline{c}_{(1)}) \otimes \underline{b}_{(2)}^{(\bar{2})} \underline{c}_{(2)}^{(\bar{1})} \otimes \underline{b}_{(2)}^{(\bar{2})} \otimes \underline{c}_{(2)}^{(\bar{2})} = (\text{id} \otimes \Delta_H) \underline{\Delta}_L(b \otimes c)\end{aligned}$$

where Δ_H is the left tensor product coaction of H on $B \otimes B$. We used the comodule property for the H -coaction for the third equality. Therefore we have just found that

$$(\text{id} \otimes \Delta_H) \underline{\Delta}_L \beta_U(h) = 1 \otimes h_{(1)} S h_{(3)} \otimes \beta_U(h_{(2)}). \quad (18)$$

Applying $\text{id}_B \otimes \epsilon_H \otimes \text{id}_B \otimes \text{id}_B$ to both sides of (18) we find $\underline{\Delta}_L \beta_U(h) = 1 \otimes \beta_U(h)$, i.e. β_U is left-invariant with respect to the (braided) left coaction of B . Using this left-invariance we can compute (18) further to find

$$1 \otimes h_{(1)} S h_{(3)} \otimes \beta_U(h_{(2)}) = (\text{id} \otimes \Delta_H) \underline{\Delta}_L \beta_U(h) = (\text{id} \otimes \Delta_H)(1 \otimes \beta_U(h)) = 1 \otimes \Delta_H(\beta_U(h)),$$

which is the required intertwiner property of β_U .

Conversely, assume that $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 B$ is left B -invariant and an intertwiner for the left adjoint coaction of H . One then immediately finds for any $h \in \ker \epsilon_H$

$$\Delta_L(\beta_U(h)) = (\text{id} \otimes \Delta_H) \underline{\Delta}_L \beta_U(h) = 1 \otimes h_{(1)} S h_{(3)} \otimes \beta_U(h_{(2)}).$$

Using this fact one computes

$$\begin{aligned}\Delta_L(\omega_U(h)) &= \Delta_L(j(S h_{(1)}) \beta_U(\pi_\epsilon(h_{(2)})) j(h_{(3)}) + S j(h_{(1)}) d_U j(h_{(2)})) \\ &= 1_B \otimes S h_{(2)} h_{(3)} S h_{(5)} h_{(6)} \otimes j(S h_{(1)}) \beta_U(\pi_\epsilon(h_{(4)})) j(h_{(7)}) + 1_{B \bowtie H} \otimes S j(h_{(1)}) d_U j(h_{(2)}) \\ &= 1_{B \bowtie H} \otimes j(S h_{(1)}) \beta_U(\pi_\epsilon(h_{(2)})) j(h_{(3)}) + 1_{B \bowtie H} \otimes S j(h_{(1)}) d_U j(h_{(2)}) \\ &= 1_{B \bowtie H} \otimes \omega_U(h),\end{aligned}$$

so the connection corresponding to β_U is left $B \bowtie H$ -invariant.

Similar arguments as in the proof of Proposition 3.5 show that any left B -invariant $\beta_U : \ker \epsilon_H \rightarrow \Omega^1 B$ can be expressed in the form $\beta_U(h) = \underline{S} \gamma(h) \underline{d}_U \gamma(h) \underline{d}_U$, where $\gamma : \ker \epsilon_H \rightarrow \ker \epsilon$

is given by $\gamma = (\underline{\epsilon} \otimes \text{id}) \circ \beta_U$. Since β_U is an intertwiner for the left adjoint coaction of H we have

$$\gamma(h)_{\underline{(1)}}^{(\bar{1})} \gamma(h)_{\underline{(2)}}^{(\bar{1})} \otimes \underline{S} \gamma(h)_{\underline{(1)}}^{(\bar{2})} d_U \gamma(h)_{\underline{(2)}}^{(\bar{2})} = h_{(1)} S h_{(3)} \otimes \underline{S} \gamma(h_{(2)})_{\underline{(1)}} d_U \gamma(h_{(2)})_{\underline{(2)}},$$

where we used that \underline{S} is a left H -comodule map. Due to the form of d_U , the above equality is in $H \otimes B \otimes B$. Applying $\underline{\epsilon}$ to the middle factor and using the fact that B is an H -comodule coalgebra one finds

$$\Delta_H(\gamma(h)) = h_{(1)} S h_{(3)} \otimes \gamma(h_{(2)}),$$

i.e. the required intertwiner property. Conversely, given $\gamma : \ker \epsilon_H \rightarrow \ker \underline{\epsilon}$ which is an intertwiner for the left adjoint coaction we find

$$\begin{aligned} \Delta_H(\beta_U(h)) &= \Delta_H(\underline{S} \gamma(h)_{\underline{(1)}} d_U \gamma(h)_{\underline{(2)}}) \\ &= \gamma(h)_{\underline{(1)}}^{(\bar{1})} \gamma(h)_{\underline{(2)}}^{(\bar{1})} \otimes \underline{S} \gamma(h)_{\underline{(1)}}^{(\bar{2})} d_U \gamma(h)_{\underline{(2)}}^{(\bar{2})} \\ &= \gamma(h)^{(\bar{1})} \otimes \underline{S} \gamma(h)^{(\bar{2})}_{\underline{(1)}} d_U \gamma(h)^{(\bar{2})}_{\underline{(2)}} \\ &= h_{(1)} S h_{(3)} \otimes \underline{S} \gamma(h_{(2)})_{\underline{(1)}} d_U \gamma(h_{(2)})_{\underline{(2)}} = h_{(1)} S h_{(3)} \otimes \beta_U(h_{(2)}), \end{aligned}$$

as required. Finally, if the map β_U is expressed in terms of the map γ then the canonical splitting i corresponding to ω_U and given by $i = (\epsilon \otimes \text{id}) \circ \omega_U$ comes out as stated in the proposition. \square

Notice that the map $\gamma : \ker \epsilon_H \rightarrow \ker \underline{\epsilon}$ defined in Proposition 6.6 can be uniquely extended to the map $\gamma : H \rightarrow B$ by requiring $\gamma(1) = 1$. Then $\underline{\epsilon} \circ \gamma = \epsilon$ and $i = (\gamma \otimes \text{id}) \circ \Delta$. Therefore for these β_U we are in the setting of Proposition 3.5 or Example 3.7 for the stated map i . The smallest horizontal right ideal is

$$\mathcal{Q}_0 = \text{span}\{\gamma(q_{(1)})q_{(2)} \triangleright b \otimes q_{(3)}h - \underline{\epsilon}(b)\gamma(q_{(1)}h_{(1)}) \otimes q_{(2)}h_{(2)} \mid q \in \mathcal{Q}, b \in B, h \in H\}.$$

We see that a choice of left-invariant ω_U , \mathcal{Q}_{hor} and left-covariant $\Omega^1(B)$ yields a suitably left-invariant $\Omega^1(B \rtimes H)$.

Example 6.7 Let $P = B \rtimes H$ be viewed as a quantum principle bundle as above with trivialisation $j(h) = 1 \otimes h$. Let γ obey the condition in Proposition 6.6 and let $\Omega^1(B)$ be braided left

B -covariant and H -covariant. Then P has a natural left-covariant calculus $\Omega^1(B \rtimes H)$ such that $\Omega_{\text{hor}}^1 = P(dB)P$ and

$$\omega(h) = j(Sh_{(1)})\beta(\pi_\epsilon(h_{(2)}))j(h_{(3)}) + j(Sh_{(1)})dj(h_{(2)})$$

for $\beta : \ker \epsilon \rightarrow \Omega_P^1(B)$ defined by $\beta(h) = \underline{S}\gamma(h)_{(1)}d\gamma(h)_{(2)}$ is a connection on it. Here $\Omega_P^1(B) = \pi_{\mathcal{N}}(\Omega^1 B)$, where $\pi_{\mathcal{N}} : \Omega^1 P \rightarrow \Omega^1(P)$ is the canonical surjection.

Proof We take $\mathcal{Q}_{\text{hor}} = \langle \mathcal{Q}_0, \mathcal{Q}_B P \rangle$ corresponding to $\mathcal{N}_{\text{hor}} = \langle \mathcal{N}_0, P\mathcal{N}_B P \rangle$ as in Example 3.6. Since $\Delta_L(b \otimes c) = (\text{id} \otimes \Delta_H)\underline{\Delta}_L(b \otimes c)$ for any $b \otimes c \in B \otimes B$ viewed inside $(B \rtimes H)^{\otimes 2}$ on the left hand side, it is clear that if $\Omega^1(B)$ is defined by \mathcal{N}_B which is both $\underline{\Delta}_L$ and Δ_H covariant then it is covariant under the Δ_L coaction of $B \rtimes H$. We then use the preceding Proposition 6.6 to establish that $\omega_U(h) = j(Sh_{(1)})\beta_U(\pi_\epsilon(h_{(2)}))j(h_{(3)}) + j(Sh_{(1)})d_U j(h_{(2)})$, where $\beta_U(h) = \underline{S}\gamma(h)_{(1)}d_U \gamma(h)_{(2)}$ is a left-invariant connection. We extend γ to the whole of H by setting $\gamma(1) = 1$. Then the corresponding splitting is $i(h) = \gamma(h_{(1)}) \otimes h_{(2)}$ and we construct a left-covariant calculus $\Omega^1(B \rtimes H)$ by taking $\mathcal{Q}_{B \rtimes H} = \langle \mathcal{Q}_{\text{hor}}, i(\mathcal{Q})(B \rtimes H) \rangle = \text{span}\{q_B b \otimes h, \gamma(q_{(1)})q_{(2)} \triangleright b \otimes q_{(3)}h \mid q \in \mathcal{Q}, q_B \in \mathcal{Q}_B, b \in B, h \in H\}$, as in Proposition 3.5. \square

Bosonisation may be viewed as a special kind of biproduct, albeit originating[30] from other considerations than [31]. Here H is a dual quasitriangular Hopf algebra[39]cf[40] and B a braided group in its category of (say) left comodules. It has a bosonisation $B \rtimes H$ where the required action is induced by evaluating against the dual quasitriangular structure, see [16] and cf[1] (where the example of the quantum double as a bundle was emphasised). Explicitly,

$$(b \otimes h)(c \otimes g) = bc^{(\bar{2})} \otimes h_{(2)}g\mathcal{R}(c^{(\bar{1})} \otimes h_{(1)}), \quad \Delta(b \otimes h) = b_{(1)} \otimes b_{(2)}^{(\bar{1})}h_{(1)} \otimes b_{(2)}^{(\bar{2})} \otimes h_{(2)}$$

where $\Delta_L b = b^{(\bar{1})} \otimes b^{(\bar{2})}$ is the coaction of H (summation understood).

To give a concrete application, we take $B = \underline{H}$ to be a braided version of H obtained by transmutation[39]. \underline{H} is then a left H -module coalgebra with the coaction provided by left adjoint coaction, i.e. $\Delta_H(h) = h_{(1)}Sh_{(3)} \otimes h_{(2)}$. It is clear that the map $\gamma : \underline{H} \rightarrow H$, $\gamma(h) = h$ satisfies all the requirements of Proposition 6.6 therefore we have

Proposition 6.8 *Let $B = \underline{H}$, $P = \underline{H} \rtimes H$ and $\gamma = \text{id}$. Let the braided left-covariant calculus on \underline{H} be generated by $\mathcal{Q}_{\underline{H}} \subset \ker \underline{\epsilon}$. Assume that the ideal $\mathcal{Q} \subset \ker \epsilon_H$ is generated by $\{q_i\}_{i \in I}$.*

Then the corresponding right ideal $\mathcal{Q}_P \subset \ker \epsilon_P$ defining left-covariant calculus on $P = \underline{H} \bowtie H$ as in Example 6.7 is generated by $\{\Delta q_i\}_{i \in I}$ and the generators of $\mathcal{Q}_{\underline{H}}$. The induced calculus $\Omega_P^1(\underline{H})$ is generated by $\langle \mathcal{Q}, \mathcal{Q}_{\underline{H}} \rangle$.

Proof The braided product \cdot in \underline{H} is related to the original product in H by

$$gh = g_{(1)} \cdot h_{(2)} \mathcal{R}(h_{(1)} S h_{(3)} \otimes g_{(2)}).$$

Since γ is an identity map one finds the corresponding splitting $i = \Delta$. For any $g, h \in \ker \epsilon$ we find

$$\begin{aligned} i(g)i(h) &= (g_{(1)} \otimes g_{(2)})(h_{(1)} \otimes h_{(2)}) = g_{(1)} \cdot h_{(2)} \mathcal{R}(h_{(1)} S h_{(3)} \otimes g_{(2)}) \otimes g_{(3)} h_{(2)} \\ &= g_{(1)} h_{(1)} \otimes g_{(2)} h_{(2)} = i(gh). \end{aligned}$$

This implies that if \mathcal{Q} is generated by $\{q_i\}_{i \in I}$ as a right ideal in H then $\mathcal{Q}_P = \langle \mathcal{Q}_{\underline{H}} P, i(\mathcal{Q}) P \rangle$ is generated by generators of $\mathcal{Q}_{\underline{H}}$ and $\{i(q_i)\}_{i \in I}$ as a right ideal in P . Since i is the same as the coproduct in H , the assertion follows.

To derive the induced calculus on \underline{H} first note that $\ker \epsilon_P = \ker \epsilon \otimes 1 \oplus \underline{H} \otimes \ker \epsilon_H$, where the splitting is given by the projection $\Pi : \ker \epsilon_P \rightarrow \ker \epsilon \otimes 1$, $\Pi = \pi_\epsilon \otimes \epsilon$. The differential structure on \underline{H} is determined by the image of \mathcal{Q}_P under this projection. Clearly $\Pi(\mathcal{Q}_{\underline{H}} P) = \mathcal{Q}_{\underline{H}}$. Furthermore, for any $b \in \underline{H}, h \in H, q \in \mathcal{Q}$ we find

$$\begin{aligned} \Pi((q_{(1)} \otimes q_{(2)})(b \otimes h)) &= \Pi(q_{(1)} \cdot b_{(2)} \mathcal{R}(b_{(1)} S b_{(3)} \otimes q_{(2)}) \otimes q_{(3)} h) \\ &= \Pi(q_{(1)} b \otimes q_{(2)} h) = \pi_\epsilon(qb) \epsilon(h) = qb \epsilon(h). \end{aligned}$$

Therefore \mathcal{Q}_P restricted to \underline{H} coincides with $\langle \mathcal{Q}_{\underline{H}}, \mathcal{Q} \rangle$ as stated. \square

In particular, if $\mathcal{Q}_{\underline{H}}$ in the preceding proposition is chosen to be trivial and thus the calculus on \underline{H} to be the universal one, the induced calculus is non-trivial and is a braided version of the calculus on H . Notice also that since [10]

$$\underline{H} \bowtie H \cong H \bowtie H \cong D(H)^*,$$

(the latter in the case where \mathcal{R} is factorisable) the above construction gives a natural left-covariant differential structure on the double cross products $H \bowtie H$ (see [6]) and the duals of the

Drinfeld double $D(H)$ (see [40]). For example, if $H = A(R)$, is a matrix quantum group corresponding to a regular solution of the quantum Yang-Baxter equation (suitably combined with q -determinant or other relations) then $\underline{H} = B_L(R)$, the left handed version of the corresponding matrix braided group[41]. The latter is generated by the matrix \mathbf{u} subject to the left-handed braided matrix relations

$$R\mathbf{u}_1 R_{21}\mathbf{u}_2 = \mathbf{u}_2 R\mathbf{u}_1 R_{21}$$

and suitable braided determinant or other relations. If \mathbf{t} denotes the matrix of generators of $A(R)$ then the cross relations in $B_L(R) \bowtie A(R)$ are given by

$$\mathbf{t}_1 \mathbf{u}_2 = R_{21} \mathbf{u}_2 R_{21}^{-1} \mathbf{t}_1$$

as the left-handed version of the formulae in [7]. The isomorphism with $A(R) \bowtie A(R)$ as generated by \mathbf{s} and \mathbf{t} say (and the cross relations $R\mathbf{s}_1 \mathbf{t}_2 = \mathbf{t}_2 \mathbf{s}_1 R$) is $\mathbf{s} = \mathbf{u}\mathbf{t}$ and the \mathbf{t} generators identified, as the appropriate left-handed version of [6]. In particular, taking R to be the standard $SU_q(2)$ R-matrix and $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the relations for $BSU_q(2) = \underline{SU_q(2)}$ come out as

$$da = ad, \quad cd = q^2 dc, \quad db = q^2 bd, \quad bc = cb + (q^{-2} - 1)(a - d)d,$$

$$ac = ca + q^{-2}(1 - q^{-2})cd, \quad ab = ba + (q^{-2} - 1)bd, \quad ad - q^2 bc = 1$$

The corresponding bosonisation $BSU_q(2) \bowtie SU_q(2)$ is isomorphic to the quantum Lorentz group $SU_q(2) \bowtie SU_q(2)$ and thus Proposition 6.8 allows one to construct a differential calculus on the quantum Lorentz group. Moreover, as explained in [7], the bosonisation form of the quantum double is quite natural if we would like to regard it as a q -deformed quantum mechanical algebra of observables or ‘quantum phase space’. It is therefore natural to build its differential calculus from this point of view.

Finally, one has braided covector spaces $V^*(R', R)$ in the category of left $A(\tilde{R})$ -comodules, with additive braided group structure. Here $A(\tilde{R})$ denotes a dilatonic extension. The bosonisation $V^*(R', R) \bowtie A(\tilde{R})$ has been introduced in [6] as a general construction for inhomogeneous quantum groups such as the dilaton-extended q -Poincaré group $\mathbb{R}_q^n \bowtie SO_q^{\tilde{}}(n)$. The detailed construction of the required intertwiner map γ in this case will be addressed elsewhere. We note only that in the classical $q = 1$ case it can be provided by a map $\gamma : \mathbb{R}^n \rightarrow SO(n)$ such that

$\gamma(g.x) = g\gamma(x)g^{-1}$ for all $g \in SO_n$. For example, for $n = 3$ the map $\gamma(x) = \exp(x)$ has this property, where x is viewed in so_3 by the Pauli matrix basis and exponentiated in $SO(3)$. The q -deformed version of such maps should then allow the application of the above methods to obtain natural left-invariant calculi on inhomogeneous quantum groups as well.

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